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The complexity of partitioning into disjoint cliques and a triangle-free graph

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To Nina (Pascal's daughter, born on Nov 1 2015) and Ilyas (Marin's son, born on Nov 6 2015)

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ABSTRACT

Motivated by Chudnovsky's structure theorem of bull-free graphs, Abu-Khzam, Feghali, and Müller have recently proved that deciding if a graph has a vertex partition into disjoint cliques and a triangle-free graph is NP-complete for five graph classes. The problem is trivial for the intersection of these five classes. We prove that the problem is NP-complete for the intersection of two subsets of size four among the five classes. We also show NP-completeness for other small classes, such as graphs with maximum degree 4 and line graphs.

1. Introduction

In this paper we consider the problem of recognizing graphs having a vertex partition into disjoint cliques and a triangle-free graph. We say that a graph is *partitionable* if it has such a partition. The vertices in the P_3-free part are colored blue and the vertices in the K_3-free part are colored red. This problem is known to be NP-complete on general graphs [5]. The NP-completeness on bull-free graphs was motivated by an open question in [11] (after Thm 2.1) about the complexity of recognizing the class τ_1 introduced by Chudnovsky [3] in her characterization of bull-free graphs. Abu-Khzam, Feghali, and Müller [1] have then investigated the complexity of deciding whether a bull-free graph is partitionable. They have shown the following.

Theorem 1 ([1]). *Recognizing partitionable graphs is NP-complete even when restricted to the following classes:*

(1) *planar graphs,*
(2) *K_4-free graphs,*
(3) *bull-free graphs,*
(4) *$(C_5, \ldots, C_t)$-free graphs (for any fixed t),*
(5) *perfect graphs.*

In Section 2, we prove Theorem 2 which improves Theorem 1. The classes h_1 and h_2 of Theorem 2 show that the problem remains NP-complete for the intersection of two subsets of size four among the five classes of Theorem 1 (graphs in the intersection of the five classes are partitionable). The class h_1 also answers the open question [1] of the complexity

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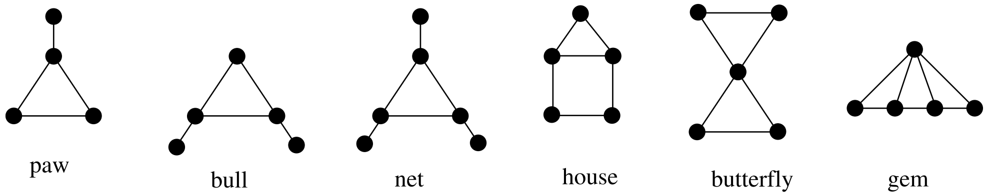

Fig. 1. Some small graphs and their name.

of recognizing partitionable Meyniel graphs, since every graph in h_1 is a parity graph and parity graphs correspond to gem-free Meyniel graphs. We also show NP-completeness for several other classes. The classes h_3 to h_9 are motivated by the introduction of other natural forbidden induced subgraphs (mainly C_4, K_4^-, and $K_{1,3}$) and/or restriction on the maximum degree. The interesting feature of every class is briefly discussed at the end of its dedicated subsection. We study the tightness of this result in Section 3 by considering all the intersections between every two graph classes of Theorem 2.

We use standard notations for graphs (see [12]), some of them are reminded in Fig. 1. For interpretation of the references to color in the other figures, the reader is referred to the web version of this article.

A *k-vertex* is a vertex of degree k. Given a graph G, we denote its line graph by $L(G)$. Given a graph class $\mathcal{C}$, we denote by $L(\mathcal{C})$ the set of line graphs of graphs in $\mathcal{C}$.

2. Main result

In this section we prove the following result.

Theorem 2. *Recognizing partitionable graphs is NP-complete even when restricted to the following classes:*

h_1: *planar* ($C_4, \ldots, C_t$, *bull, gem, odd hole)-free graphs with maximum degree* 8,
h_2: *planar* (K_4, *bull, house,* $C_5, \ldots, C_t$)*-free graphs,*
h_3: *planar* (K_4, C_4, *gem,* $C_7, \ldots, C_t$, *odd hole of length* $\geqslant 7$)*-free graphs with maximum degree* 7,
h_4: (K_4, $C_5, \ldots, C_t$, *net, odd hole)-free graphs with maximum degree* 8,
h_5: (K_4^-, *butterfly,* $C_6, \ldots, C_t$)*-free graphs with maximum degree* 4,
h_6: (K_4, K_4^-, $C_4, \ldots, C_t$, *butterfly)-free graphs,*
h_7: *planar* ($K_{1,3}$, K_4^-, $C_4, \ldots, C_t$, *odd hole)-free graphs with maximum degree* 6,
h_8: *planar* ($K_{1,3}$, K_4^-, $C_9, \ldots, C_t$, *odd hole)-free graphs with maximum degree* 5,
h_9: ($K_{1,3}$, K_4^-, $C_4, \ldots, C_t$, K_5, *odd hole)-free graphs with maximum degree* 5.

Kratochvíl proved that PLANAR (3, $\leqslant$4)-SAT is NP-complete [6]. In this restricted version of SAT, the graph of variable-clause incidences of the input formula must be planar, every clause is a disjunction of exactly three literals, and every variable occurs in at most four clauses. For every class considered in Theorem 2, we provide a reduction from PLANAR (3, $\leqslant$4)-SAT. Given an instance formula I of PLANAR (3, $\leqslant$4)-SAT, we construct a graph G such that G is partitionable if and only if I is satisfiable.

For the classes h_1, h_2, h_3, and h_4, the boolean value true is associated to the color red, the boolean value false is associated to the color blue, and the clause gadget is P_3. This way, an unsatisfied clause corresponds to a blue P_3. For the classes h_5 and h_6, the boolean value true is associated to the color blue, the boolean value false is associated to the color red, and the clause gadget is K_3. This way, an unsatisfied clause corresponds to a red K_3. For brevity, we say that a vertex with the color associated to the boolean value true (resp. false) is colored true (resp. false).

Given a variable x, a *variable gadget* is a graph G_x with two disjoint subsets of vertices S_x and $S_{\overline{x}}$ such that:

- There exists an involutive automorphism of G_x which swaps S_x and $S_{\overline{x}}$.
- There exists a partition of G_x such that every vertex in S_x is colored true and no blue vertex in $S_x \cup S_{\overline{x}}$ is adjacent to a blue vertex.
- No partition of G_x is such that both a vertex in S_x and a vertex in $S_{\overline{x}}$ are colored true.

The variable gadget depends on the considered graph class and is built on *forcers*. A forcer is a partitionable graph with a specified vertex q.

- A red forcer is such that q is red in every partition.
- A blue forcer is such that q is blue in every partition and there exists a partition such that every neighbor of q is red.

We construct G from I as follows. We take one copy of the variable gadget per variable. We take one copy of the clause gadget (either P_3 or K_3) per clause. Each of the 3 vertices of the clause gadget corresponds to a literal of the clause. The vertices in $S_x \cup S_{\overline{x}}$ are depicted in green in the representation of the variable gadgets (Figs. 4 and 8). A subset of these green vertices corresponds to the literals of the variable x. For every literal ℓ_x of I, one vertex corresponding to ℓ_x in G_x is identified with the vertex corresponding to ℓ_x in the clause gadget.

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