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Combinatorial and spectral properties of König–Egerváry graphs

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ABSTRACT

Some combinatorial and spectral properties of König–Egerváry (K–E) graphs are presented. In particular, some new combinatorial characterizations of K–E graphs are introduced, the Laplacian spectrum of particular families of K–E graphs is deduced, and a lower and upper bound on the largest and smallest adjacency eigenvalue, respectively, of a K–E graph are determined.

1. Introduction

For every graph H of order n, $n = \alpha(H) + \tau(H)$, where $\alpha(H)$ is the independence number of H and $\tau(H)$ is the vertex cover number of H (also called the transversal number of H, which is the cardinality of a smallest set of vertices meeting all the edges of H). Furthermore, since $\tau(H) \geq \mu(H)$, where $\mu(H)$ is the matching number it follows that $\alpha(H)+\mu(H) \leq n$ (the definitions of independence and matching numbers are given in Section 2). A graph G of order n such that $\alpha(G) + \mu(G) = n$ is called a *König–Egerváry graph*, or simply *K–E graph* [15,27]. Therefore, since for every graph G of order n, $\mu(G) \leq n/2$, from the definition, it is immediate that if G is K–E, then $\alpha(G) \geq n/2$. Observe that K–E graphs can be also characterized as the graphs where the matching and vertex cover numbers are equal. Notice also that every bipartite graph has this property by König–Egerváry's theorem.

K–E graphs had different names in several publications. For instance, they are called *matching-covered* graphs in [25], where some characterizations were introduced. The combinatorial properties of K–E graphs have been intensively studied from different points of view (see for instance [19,22–24]). However, according to our knowledge, so far none spectral property of these graphs have been published. In this paper, besides the presentation of some new combinatorial properties of K–E graphs a few spectral properties are deduced.

In Section 2, the notation and the main basic concepts and results in this context are presented. Section 3 is devoted to the introduction of new combinatorial properties of K–E graphs, namely some combinatorial characterizations of these graphs and their properties when they have complete dominating induced matchings are introduced. In Section 4 the spectral Laplacian properties of particular families of K–E graphs are analyzed and some bounds for the largest and smallest adjacency eigenvalues of arbitrary K–E graphs are determined.

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2. Preliminaries

Throughout this paper we consider undirected finite simple graphs (that is, loopless and without multiple edges) G with vertex set $V(G)$ and edge set $E(G)$. If $X = \{v_1, \ldots, v_p\} \subseteq V(G)$, then $\langle v_1, \ldots, v_p \rangle_G$ (or $\langle X \rangle_G$) is the subgraph of G induced by the vertex subset X, that is, obtained from G by deleting the vertices in $Y = V(G) \setminus X$. This graph is also denoted $G - Y$.

A subset $S \subseteq V(G)$ is *independent* if no two vertices in S are adjacent. An independent set of maximum cardinality will be referred to as a *maximum independent set* of G and its cardinality is designated *independence number* of G and denoted $\alpha(G)$. A *matching* M of a graph G is an edge subset such that there are no two edges in M with a common end-vertex. The set of the vertices of the edges of a matching M is denoted $V(M)$. A matching with maximum cardinality is designated a *maximum matching* and its cardinality is denoted $\mu(G)$. A matching which cover all vertices in $V(G)$ is called a *perfect matching*. It is immediate that a perfect matching is a maximum matching but the converse is not necessarily true.

The *adjacency matrix* of a graph G of order n is the $n \times n$ symmetric matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if $ij \in E(G)$ and $a_{ij} = 0$ otherwise. The *Laplacian matrix* of G is $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees of G. The matrices $A(G)$ and $L(G)$ are all real and symmetric. From Geršgorin's theorem, it follows that the eigenvalues of $L(G)$ are nonnegative real numbers. The spectrum of a matrix C is denoted $\sigma(C) = \{\gamma_1^{[i_1]}, \ldots, \gamma_s^{[i_s]}\}$, where $\gamma_j^{[i_j]}$ means that γ_j is an eigenvalue of the matrix C with multiplicity i_j, for $j = 1, \ldots, s$. In the particular cases of the matrices $A(G)$ and $L(G)$, their spectra are denoted by $\sigma_A(G)$ and $\sigma_L(G)$, respectively. Let us to say that the eigenvalues of $A(G)$ and of $L(G)$ respectively are the *adjacency eigenvalues* of G and the *Laplacian eigenvalues* of G. The second-smallest eigenvalue of the Laplacian matrix of $L(G)$ was called by Fiedler [17] the *algebraic connectivity* of G.

For further details about basic concepts and notation the reader is referred to [12,13].

Before to proceed to the next section it is worth to recall the following well known result about maximum independent sets.

Theorem 2.1 ([2]). *An independent set S of a graph G is a maximum independent set if and only if every independent set I disjoint from S is matched into S.*

As an immediate consequence, if S is a maximum independent set of a graph G of order n and $I_1, \ldots, I_k$ are maximal independent sets which jointly with S form a partition of $V(G)$, applying Theorem 2.1, it follows that every matching $M \subseteq E(G)$ has cardinality not greater than $n - \alpha(G)$ and thus

$$\alpha(G) + \mu(G) \leq n. \tag{2.1}$$

According to [3], for every graph G of order n, $\alpha(G) + 2\mu(G) \geq n$. Therefore, combining this inequality with (2.1), we obtain

$$\alpha(G) + \mu(G) \leq n \leq \alpha(G) + 2\mu(G). \tag{2.2}$$

Notice that if a graph G has a maximum independent set S such that $G - S$ has a perfect matching which is a maximum matching for G, then the right inequality in (2.2) holds as equality.

Example 2.1. The next figure depicts a graph G of order 6 for which the right inequality in (2.2) holds as equality. For instance, the vertex subset $S = \{1, 6\}$ is a maximum independent set and $G - S$ has the perfect matching $M = \{23, 45\}$ which is a maximum matching for G.

In this paper we focus on K–E graphs, that is, on graphs G of order n such that $\alpha(G) + \mu(G) = n$ [15,27].

3. Combinatorial properties

3.1. Combinatorial characterizations of K–E graphs

The next theorem gives a combinatorial characterization of K–E graphs. For a matching M of a graph, let $V(M)$ denote the set of end-vertices of the edges in M; notice that $|V(M)| = 2|M|$.

Theorem 3.1. *Let G be a graph of order n. Then G is K–E if and only if for any maximum independent set $S \subset V(G)$ and for any maximum matching $M \subseteq E(G)$ one has*

$$V(G) = V(M) \cup S \quad \text{and} \tag{3.1}$$

$$\forall xy \in M, \quad \{x, y\} \cap S \neq \emptyset. \tag{3.2}$$

Proof. Consider a maximum independent set $S \subset V(G)$ and a maximum matching $M \subseteq E(G)$.

- Let us assume that (3.1) and (3.2) hold. Then it is immediate that $|S| + |M| = n$.
- Conversely, let us assume that (3.1) or (3.2) does not hold.
  (1) If (3.1) does not hold, then $\exists v \in V(G) \setminus (V(M) \cup S)$ and thus $|S| + |M| \leq |V(M) \cup S| < |V(G)|$.
  (2) If (3.2) does not hold then $\exists xy \in M$ such that $\{x, y\} \cap S = \emptyset$. Therefore $|S| + |M| < |S \cup V(M)| \leq |V(G)|$. $\square$