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Total domination in maximal outerplanar graphs

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ABSTRACT

We show that the total domination number of a maximal outerplanar graph G is bounded above by $\frac{n+k}{3}$, where n is the order of G and k is the number of vertices of degree 2. For $k > \frac{n}{3}$, a better bound is given by $\frac{2(n-k)}{3}$. For $k > \frac{n}{3}$, we improve the upper bound of $\frac{n+k}{4}$ on the usual domination number.

1. Introduction

The graphs we consider in this paper are undirected and simple. Given a graph $G = (V, E)$ with $v \in V$, the *open neighborhood* of v, denoted $N(v)$, is defined as the set of all vertices adjacent to v. The *closed neighborhood* of v, denoted $N[v]$, is the union of $N(v)$ and $\{v\}$. A set of vertices D is a *dominating set* if for every $u \in V$, there exists $v \in D$ such that $u \in N[v]$. The *domination number* of a graph G, denoted by $\gamma(G)$, is the minimum size of a dominating set of vertices in G. If, for every $u \in V$, there is a $v \in D$ such that $u \in N(v)$, then D is a *total dominating set* of G. The minimum cardinality of such a set is denoted by $\gamma_t(G)$.

The question of determining the domination number for a graph is a well known NP-hard problem. Bounds for the domination numbers have been found for special classes of graphs [3]. Planar graphs have been studied in [5,6].

A planar graph is *outerplanar* if it can be embedded in the plane so that all its vertices lie on the same face; hereafter we assume this face to be *exterior* or the *outer* face. An outerplanar graph is *nonseparable* if it has a plane representation with a hamiltonian face. For nonseparable outerplanar graphs the hamiltonian face is unique [8]. In [7] it was proved that if G is a nonseparable outerplanar graph, then

$$\left\lceil \frac{2|V| - |E|}{3} \right\rceil \leq \gamma(G) \leq \left\lceil \frac{|V|}{3} \right\rceil.$$

Note that a maximal outerplanar graph is nonseparable. Campos and Wakabayashi showed in [1] that if G is an n-vertex maximal outerplanar graph, then $\gamma(G) \leq \frac{n+k}{4}$ where k is the number of vertices of degree 2 in G. By using a simple coloring method, Tokunaga proved the same result independently in [9]. Li, Zhu, Shao and Xu (in [4]) improved the latter result by showing that $\gamma(G) \leq \frac{n+k}{4}$, where k is the number of pairs of consecutive 2-degree vertices with distance at least 3 on the outer cycle.

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In this paper we show that if G is a maximal outerplanar graph of order n with k vertices of degree 2, then $\gamma_t(G) \leq \frac{n+k}{3}$. We further show that these upper bounds on γ and γ_t can be improved if $k > \frac{n}{3}$.

Some terminology we will use throughout this paper is: A t-vertex is a vertex of degree t. Given an embedding of an outerplanar graph, an *outer edge* will be an edge on the outer face. Moreover, $d(u, v)$ will denote the distance between the vertices u and v. For any undefined notation or terminology the reader is referred to [2].

2. Total domination

We now present the proof of our main result. The reader can check that for $k > \frac{n}{3}$, it holds that $\frac{2(n-k)}{3}$ is less than $\frac{n+k}{3}$. We observe that every maximal outerplanar graph has at least two but not more than $n/2$ vertices of degree 2, i.e., $2 \leq k \leq n/2$.

Theorem 1. *If G is a maximal outerplanar graph of order $n \geq 3$ with k vertices of degree 2, then*

$$\gamma_t(G) \leq \begin{cases} \dfrac{2(n-k)}{3}, & \text{if } k > \dfrac{n}{3} \text{ and } n \geq 5 \\ \dfrac{n+k}{3}, & \text{otherwise.} \end{cases}$$

Proof. Both bounds hold for all possible k, but which is the better bound depends on k. We first prove that $\gamma_t(G) \leq \frac{n+k}{3}$ for all such G. The proof is by induction on n. If $n = 3$ the result clearly holds, so let G be a maximal outerplanar graph of order $n > 3$ with k vertices of degree 2 and suppose that for any maximal outerplanar graph G' of order $n' < n$ with k' vertices of degree 2 we have $\gamma_t(G') \leq \frac{n'+k'}{3}$.

Claim 1. *We may assume that if v is any 2-vertex, then the neighbors of v have degrees 3 and 4, respectively.*

Let u and w be the neighbors of v. Since v has degree 2 and G is maximal outerplanar, u and w are adjacent. Since $n > 3$, they have a common neighbor $x \neq v$.

Suppose u and w both have degree at least 4. Let $y \neq v, x$ be such that uy is an outer edge, let $z \neq v, x$ be such that wz is an outer edge, and note that x has degree at least 4. Let G' be obtained from G by removing v and contracting edge uw to the vertex u'. Then G' is maximal outerplanar, G' has two fewer vertices than G and G' has one 2-vertex less than G, as both u' and x have degrees at least 3 in G'. By the inductive hypothesis we have $\gamma_t(G') \leq \frac{n-2+k-1}{3} = \frac{n+k}{3} - 1$. Let D' be such a total dominating set of G'. If $u' \in D'$, then $D' - \{u'\} \cup \{u, w\}$ totally dominates G and has order at most $\frac{n+k}{3}$. So suppose $u' \notin D'$, and let $t \in D'$ be a vertex adjacent to u' in G'. Then $D = \begin{Bmatrix} D' \cup \{u\} & \text{if } t \text{ is adjacent to } u \text{ in } G \\ D' \cup \{w\} & \text{if } t \text{ is adjacent to } w \text{ in } G \end{Bmatrix}$ is a total dominating set of G and has cardinality at most $\frac{n+k}{3}$.

One of u and w, say w, therefore has degree 3. If u also has degree 3 then $G = K_4 - e$ and the result holds. If u has degree 4 we are done, so suppose that u has degree at least 5.

Now if x has degree 3 then, with y as before, x has a neighbor $a \neq u, w, y$ where $au \in E(G)$. Then we let $G' = G - v$ and consider any total dominating set D' of G' of cardinality $\frac{n-1+k}{3}$. If u or w is in D', then D' also totally dominates G. If not, since w and x must be dominated by D', we must have $x \in D'$ and $a \in D'$. But then $D' - \{x\} \cup \{u\}$ totally dominates G.

Therefore x has degree at least 4. Now let $G' = G - \{v, w\}$. No 2-vertex is created while one is removed. Therefore $\gamma_t(G') \leq \frac{n+k}{3} - 1$ and any such total dominating set D' can be extended to a total dominating set of G by adding u if $u \notin D'$. □

Note that the vertices x and y in the proof above must therefore be adjacent by maximality, given that u has degree 4.

Any maximal outerplanar graph H can be associated with a tree T with maximum degree at most 3, the vertices of which correspond to triangles of H and two vertices of T being adjacent when the corresponding triangles in H have a common edge. For the remainder of the paper, we let T be the tree associated with G. Note that T does not uniquely determine G but contains useful information about the structure of G.

Claim 2. *We may assume that every leaf of T is at distance at most 4 from a 3-vertex of T.*

Suppose v is a leaf of T with no 3-vertex at distance at most 4. Using Claim 1 it follows that G is isomorphic to one of the graphs G_1 or G_2 of Fig. 1 (here the squares are to be triangulated arbitrarily and in both cases only the vertices x, y and z possibly have neighbors not shown) or $n \leq 7$. In the latter case G is a subgraph of $H - y$ where H is isomorphic to one of the graphs G_1 or G_2 of Fig. 1 where none of the vertices in $\{x, y, z\}$ have additional neighbors and the result is easily checked.

In both cases of Fig. 1, let G' be obtained from G by removing all vertices shown except x, y and z.

First consider the case when $\deg_{G'}(x) \geq 3$ and $\deg_{G'}(z) \geq 3$. Then G' is maximal outerplanar, G' has five fewer vertices than G and G' has one 2-vertex less than G. By the inductive hypothesis, G' has a total dominating set of cardinality $\frac{n-5+k-1}{3} = \frac{n+k}{3} - 2$. Then $D = \begin{Bmatrix} D' \cup \{u, w\} & \text{if } G \cong G_1 \\ D' \cup \{x, w\} & \text{if } G \cong G_2 \end{Bmatrix}$ is a total dominating set of G and has cardinality at most $\frac{n+k}{3}$.

If $\deg_{G'}(x) = \deg_{G'}(z) = 2$, then G is isomorphic to either G_1 or G_2 (with none of the vertices in $\{x, y, z\}$ having additional neighbors) and so $\gamma_t(G) \leq 3 \leq \frac{8+2}{3} = \frac{n+k}{3}$.