Discrete Applied Mathematics 217 (2017) 536–547

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

A linear kernel for planar red–blue dominating set[☆]

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ARTICLE INFO

Article history:
Received 15 September 2015
Received in revised form 9 September 2016
Accepted 27 September 2016
Available online 21 October 2016

Keywords:
Parameterized complexity
Planar graphs
Linear kernels
Red–blue domination

ABSTRACT

In the Red–Blue Dominating Set problem, we are given a bipartite graph $G = (V_B \cup V_R, E)$ and an integer k, and asked whether G has a subset $D \subseteq V_B$ of at most k "blue" vertices such that each "red" vertex from V_R is adjacent to a vertex in D. We provide the first explicit linear kernel for this problem on planar graphs, of size at most $43k$.

1. Introduction

Motivation. The field of parameterized complexity (see [7,8,18]) deals with algorithms for decision problems whose instances consist of a pair (x, k), where k is known as the *parameter*. A fundamental concept in this area is that of *kernelization*. A kernelization algorithm, or *kernel*, for a parameterized problem takes an instance (x, k) of the problem and, in time polynomial in $|x| + k$, outputs an equivalent instance (x', k') such that $|x'|, k' \leq g(k)$ for some function g. The function g is called the *size* of the kernel and may be viewed as a measure of the "compressibility" of a problem using polynomial-time preprocessing rules. A natural problem in this context is to find polynomial or linear kernels for problems that admit such kernelization algorithms.

A celebrated result in this area is the linear kernel for Dominating Set on planar graphs by Alber et al. [2], which gave rise to an explosion of (meta-)results on linear kernels on planar graphs [14] and other sparse graph classes [3,9,15]. Although of great theoretical importance, these meta-theorems have two important drawbacks from a practical point of view. On the one hand, these results rely on a problem property called *Finite Integer Index*, which guarantees the *existence* of a linear kernel, but nowadays it is still not clear how and when such a kernel can be effectively *constructed*. On the other hand, at the price of generality one cannot hope that general results of this type may directly provide explicit reduction rules and small constants for particular graph problems. Summarizing, as mentioned explicitly by Bodlaender et al. [3], these meta-theorems provide simple criteria to decide whether a problem admits a linear kernel on a graph class, but finding linear kernels with reasonably small constant factors for concrete problems remains a worthy investigation topic.

☆ This work was supported by the ANR project AGAPE (ANR-09-BLAN-0159) and the Languedoc-Roussillon Project "Chercheur d'avenir" KERNEL. A preliminary short version of this work appeared in the *Proceedings of the 12th Cologne-Twente Workshop on Graphs and Combinatorial Optimization, pages 117–120, Enschede, Netherlands, May 21–23, 2013.*

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http://dx.doi.org/10.1016/j.dam.2016.09.045

Our result. In this article we follow this research avenue and focus on the Red–Blue Dominating Set problem (RBDS for short) on planar graphs. In the Red–Blue Dominating Set problem, we are given a bipartite[1] graph $G = (V_B \cup V_R, E)$ and an integer k, and asked whether G has a subset $D \subseteq V_B$ of at most k "blue" vertices such that each "red" vertex from V_R is adjacent to a vertex in D. This problem appeared in the context of the European railroad network [20]. From a (classical) complexity point of view, finding a red–blue dominating set (or rbds for short) of minimum size is NP-hard on planar graphs [1]. From a parameterized complexity perspective, RBDS parameterized by the size of the solution is $W[2]$-complete on general graphs and FPT on planar graphs [7]. It is worth mentioning that RBDS plays an important role in the theory of non-existence of polynomial kernels for parameterized problems [6].

The fact that RBDS involves a *coloring* of the vertices of the input graph makes it unclear how to make the problem fit into the general frameworks of [3,9,14,15]. In this article we provide the first explicit (and quite simple) polynomial-time data reduction rules for Red–Blue Dominating Set on planar graphs, which lead to a linear kernel for the problem.

Theorem 1. Red–Blue Dominating Set *parameterized by the solution size has a linear kernel on planar graphs. More precisely, there exists a polynomial-time algorithm that for each planar instance* (G, k)*, either correctly reports that* (G, k) *is a* No*-instance, or returns an equivalence instance* (G', k') *such that* $k' \leq k$ *and* $|V(G')| \leq 43 \cdot k'$.

This result complements several explicit linear kernels on planar graphs for other domination problems such as Dominating Set [2], Edge Dominating Set [14,19], Efficient Dominating Set [14], Connected Dominating Set [13,17], or Total Dominating Set [12]. It is worth mentioning that our constant is considerably smaller than most of the constants provided by these results. Since one can easily reduce the Face Cover problem on a planar graph to RBDS (without changing the parameter),[2] the result of Theorem 1 also provides a linear *bikernel* for Face Cover (i.e., a polynomial-time algorithm that given an input of Face Cover, outputs an equivalent instance of RBDS with a graph whose size is linear in k). To the best of our knowledge, the best existing kernel for Face Cover is quadratic [16]. Our techniques are much inspired by those of Alber et al. [2] for Dominating Set, although our reduction rules and analysis are slightly simpler. We start by describing in Section 2 our reduction rules for Red–Blue Dominating Set when the input graph is embedded in the plane, and in Section 3 we prove that the size of a reduced plane Yes-instance is linear in the size of the desired red–blue dominating set, thus proving Theorem 1. Finally, we conclude with some directions for further research in Section 4.

2. Reduction rules

In this section we propose reduction rules for Red–Blue Dominating Set, which are largely inspired by the rules that yielded the first linear kernel for Dominating Set on planar graphs [2]. The idea is to either replace the neighborhood of some blue vertices by appropriate gadgets, or to remove some blue vertices and their neighborhood when we can assume that these blue vertices belong to the dominating set. We would like to point out that our rules have also some points in common with the ones for the current best kernel for Dominating Set [4]. In Section 2.1 we present two easy elementary rules that turn out to be helpful in simplifying the instance, and then in Sections 2.2 and 2.3 we present the rules for a single vertex and a pair of vertices, respectively.

Before starting with the reduction rules, we need a definition.

Definition 1. We say that a graph G is *reduced under a set of rules* if either none of these rules can by applied to G, or the application of any of them creates a graph isomorphic to G.

With slight abuse of notation, we simply say that a graph is *reduced* if it reduced under the whole set of reduction rules that we will define, namely Rules 1, 2, 3, and 4.

We would like to point out that the above definition differs from the usual definition of *reduced* graph in the literature, which states that a graph is reduced if the corresponding reduction rules cannot be applied anymore. We diverge from this definition because, for convenience, we will define reduction rules that could be applied *ad infinitum* to the input graph, such as Case 2 of Rule 4 defined in Section 2.3. For algorithmic purposes, the reduction rules that we will define are all local and concern the neighborhood of at most 2 vertices, which is replaced with gadgets of constant size. Therefore, in order to know when a graph is reduced (see Definition 1), the fact whether the original and the modified graph are isomorphic or not can be easily checked locally in constant time.

2.1. Elementary rules

The following two elementary rules enable us to simplify an instance of RBDS. We would like to point out that similar rules have been provided by Weihe [20] in a more applied setting. We first need the definition of neighborhood.

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1 In fact, this assumption is not necessary, as if the input graph G is not bipartite, we can safely remove all edges between vertices of the same color.

2 Just consider the *radial graph* corresponding to the input graph G and its dual G^*, and color the vertices of G (resp. G^*) as red (resp. blue).