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Graphs that are simultaneously efficient open domination and efficient closed domination graphs

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ABSTRACT

A graph is an efficient open (resp. closed) domination graph if there exists a subset of vertices whose open (resp. closed) neighborhoods partition its vertex set. Graphs that are efficient open as well as efficient closed (shortly EOCD graphs) are investigated. The structure of EOCD graphs with respect to their efficient open and efficient closed dominating sets is explained. It is shown that the decision problem regarding whether a graph is an EOCD graph is an NP-complete problem. A recursive description that constructs all EOCD trees is given and EOCD graphs are characterized among the Sierpiński graphs.

1. Introduction

The *domination number*, $\gamma(G)$, of a graph G is an important classical graph invariant with many applications. It is defined as the minimum cardinality of a subset of vertices S, called *dominating set*, with the property that each vertex from $V(G) - S$ has a neighbor in S. A dominating set S of cardinality $\gamma(G)$ is called a $\gamma(G)$*-set*. The union of closed neighborhoods centered at vertices of a dominating set covers the entire vertex set. A classical question for a cover of a set is: when does this cover form a partition? A graph G is called an *efficient closed domination graph*, or *ECD graph* for short, if there exists a set P, $P \subseteq V(G)$, such that the closed neighborhoods centered at vertices of P partition $V(G)$. Such a set P is called a *perfect code* of G. More general, a set P is an r*-perfect code* of G if the r-balls centered at vertices of P partition $V(G)$.

The study of perfect codes in graphs was initiated by Biggs [5] and presents a generalization of the problem of the existence of (classical) error-correcting codes. The initial research focused on distance regular and related classes of graphs, while later the investigation was extended to general graphs, cf. [34]. To determine whether a given graph has a 1-perfect code is an NP-complete problem [3] and remains NP-complete on k-regular graphs ($k \geq 4$) [35], on planar graphs of maximum degree 3 [13,35], as well as on bipartite and chordal graphs [41]. On the positive side, the existence of a 1-perfect code can be decided in polynomial time on trees [13], interval graphs [36], and circular-arc graphs [30].

Recently the study of perfect codes in graphs was primarily focused on their existence and construction in some central families of graphs. Much research was done on standard graph products and product-like graphs [2,24,29,40,43,45].

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Among other classes of graphs on which perfect codes were investigated we mention Sierpiński graphs [8,28], cubic vertex-transitive graphs [32], circulant graphs [10], twisted tori [25], dual cubes [26], and AT-free and dually chordal graphs [4].

A graph invariant closely related to the domination number is the *total domination number* $\gamma_t(G)$ [21]. It is defined as the minimum cardinality of a subset of vertices D, called a *total dominating set*, such that each vertex from $V(G)$ has a neighbor in D. A total dominating set D of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-*set*. If we switch to neighborhoods, the union of open neighborhoods centered at vertices of a total dominating set covers the entire vertex set and again one can pose the question: when does this cover form a partition? A graph G is called an *efficient open domination graph*, or an *EOD graph* for short, if there exists a set D, $D \subseteq V(G)$, such that open neighborhoods centered at vertices of D partition $V(G)$. Such a set D is called an *EOD set*. Note that two different vertices of an EOD set are either adjacent or at distance at least 3.

The problem of deciding whether a graph G is an EOD graph is *NP*-complete [16,39]. For various properties of EOD graphs see [15], a recursive characterization of EOD trees is given in [16]. EOD graphs that are also Cayley graphs were studied in [42], while EOD grid graphs were investigated in [7,9,31]. EOD direct product graphs were characterized in [1], for other standard graph products (lexicographic, strong, disjunctive and Cartesian) see [37]. Domination-type problems studied on graph products are usually most difficult on the Cartesian product, recall the famous Vizing's conjecture [6]. It is hence not surprising that EOD graphs studied on product graphs seem to be the most difficult on the Cartesian product. For some very recent results in this direction see [33].

In this paper we study the graphs that are ECD and EOD at the same time and call them *efficient open closed domination graphs*, *EOCD graphs* for short. In the rest of the paper we shall use the term *ECD set* instead of 1-perfect code to make the notation consistent.

We proceed as follows. In the rest of this section additional definitions are given and a basic result recalled. In the next section we show how to construct an ECD graph from and EOD graph and vice versa, and consider the structure of EOCD graphs from the viewpoint of the relationship between selected EOD sets and selected ECD sets. In two extremal cases we find that for the corresponding EOCD graphs G we have $\gamma_t(G) = \gamma(G)$ and $\gamma_t(G) = 2\gamma(G)$, respectively. In Section 3 we prove that the decision problem regarding whether a graph is an EOCD graph is an NP-complete problem. On the other hand, in one of the above extremal cases, EOCD graphs can be recognized in polynomial time. Then, in Section 4, we give a recursive description of EOCD trees, while in the final section EOCD graphs are characterized among the Sierpiński graphs.

We will use the notation $[n] = \{1, \ldots, n\}$ and $[n]_0 = \{0, \ldots, n-1\}$. Throughout the article we consider only finite, simple graphs. If S is a subset of vertices of a graph, then $\langle S \rangle$ denotes the subgraph induced by S. A *matching* of a graph is an independent set of its edges. For the later use we next state the following basic result. Its first assertion has been independently discovered several times, cf. [19, Theorem 4.2], while for the second fact see [37].

Proposition 1.1. *Let G be a graph.*

(i) *If P is an* ECD *set of G, then* $|P| = \gamma(G)$.
(ii) *If D is an* EOD *set of G, then* $|D| = \gamma_t(G)$.

2. On the structure of EOCD graphs

In this section we first show that each EOD graph naturally yields an ECD graph and that each ECD graph can be modified to an EOD graph. Then we consider the structure of EOCD graphs with respect to the relationship between their EOD and ECD sets.

If D is an EOD set of a graph G, then D induces a matching M. Note that an edge from M lies in no triangle, hence its contracting produces no parallel edges. Now, let G' be the graph obtained from G by contraction all the edges from M. Then G' is an ECD graph with an ECD set consisting of the vertices obtained by the contraction of M.

Conversely, let G' be an ECD graph with an ECD set P. For every vertex $v \in P$ weakly partition the set of its neighbors arbitrarily into sets A and B. (If the degree of v is 1, then necessarily one of these sets is empty.) Let G be the graph obtained from G' by replacing every vertex $v \in P$ by two adjacent vertices v_A and v_B, and adding edges uv_A for every $u \in A$ and edges uv_B for every $u \in B$. Then G is an EOD graph with an EOD set $\{v_A, v_B : v \in P\}$.

Let G be an EOCD graph with an EOD set D and an ECD set P. Then $V(G)$ can be weakly partitioned into sets $D \cap P$, $D - P$, $P - D$, and $R = V(G) - (D \cup P)$, see Fig. 1. Clearly, some of these sets may be empty. From the definitions of ECD and EOD sets we infer the following properties.

- A vertex from $D \cap P$ (a black squared vertex in Fig. 1) can have an arbitrary number of neighbors in R, has a unique neighbor in $D - P$, and has no neighbors in $P - D$.
- A vertex from $P - D$ (a white squared vertex in Fig. 1) can have an arbitrary number of neighbors in R, a unique neighbor in $D - P$, and no neighbors in $D \cap P$.
- A vertex from $D - P$ (a black vertex in Fig. 1) can have an arbitrary number of neighbors in R and, either a unique neighbor in $P - D$ and a unique neighbor in $D - P$, or a unique neighbor in $D \cap P$.
- A vertex from R (a white vertex in Fig. 1) can have an arbitrary number of neighbors in R and either a unique neighbor in $P - D$ and a unique neighbor in $D - P$, or a unique neighbor in $D \cap P$.

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