ARTICLE IN PRESS
Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Recursive cubes of rings as models for interconnection networks

Hamid Mokhtar*, Sanming Zhou

School of Mathematics and Statistics, The University of Melbourne, Parkville, VIC 3010, Australia

ARTICLE INFO

Article history:
Received 24 February 2016
Received in revised form 6 September 2016
Accepted 12 September 2016
Available online xxxx

Keywords:
Cayley graph
Recursive cube of rings
Cube-connected cycles
Interconnection networks
Routing
Shortest path
Diameter
Wiener index
Edge-forwarding index
Vertex-forwarding index
Bisection width

ABSTRACT

We study recursive cubes of rings as models for interconnection networks. We first redefine each of them as a Cayley graph on the semidirect product of an elementary abelian group by a cyclic group in order to facilitate the study of them by using algebraic tools. We give an algorithm for computing shortest paths and the distance between any two vertices in recursive cubes of rings, and obtain the exact value of their diameters. We obtain sharp bounds on the Wiener index, vertex-forwarding index, edge-forwarding index and bisection width of recursive cubes of rings. The cube-connected cycles and cube-of-rings are special recursive cubes of rings, and hence all results obtained in the paper apply to these well-known networks.

1. Introduction

The design and analysis of interconnection networks plays an important role in parallel computing, cloud computing, VLSI, etc. In the literature, many network structures have been proposed and studied [7,2,33,28,6,25,34,17] for different purposes. Various factors need to be considered in order to achieve high performance and low construction costs of an interconnection network. Among them, vertex-transitivity, small and fixed node degree, small diameter, recursive construction, existence of efficient routing algorithms are some of the desirable properties [13,10,21,18]. For example, networks with smaller diameters will lead to shorter data transmission delay. The forwarding indices [5,12] and bisection width are also well-known measures of performance of interconnection networks [25,32,28–30,10,13].

It is widely known [12] that Cayley graphs are good models for interconnection networks due to their many desirable properties, including vertex-transitivity and efficient routing algorithms. In fact, any Cayley graph admits an all-to-all shortest path routing that loads all vertices uniformly [14], and some Cayley graphs have analogous properties with respect to edges [26,33]. In the literature, several families of Cayley graphs, including circulants, recursive circulants, hypercubes, cube-connected cycles, cube-of-rings, star graphs, butterflies and orbital regular graphs, have been studied from the viewpoint of routing algorithms [10,21,28–30], diameters, and forwarding indices [11,12,26,16,13,6,25,28–30]. All-to-all routings that uniformly load all edges along with edge-forwarding indices were given in [11] for star graphs and in [8,26,28–30,33] for a few families of Frobenius graphs.

* Corresponding author.
E-mail addresses: hmokhtar@student.unimelb.edu.au (H. Mokhtar), sanming@unimelb.edu.au (S. Zhou).

http://dx.doi.org/10.1016/j.dam.2016.09.026

Since the class of Cayley graphs is huge, it is not a surprise that not every Cayley graph has all desired network properties. For instance, the degrees of hypercubes and recursive circulants increase with their orders, and the diameters of low degree circulants are larger than the logarithm of their orders. In order to overcome shortcomings of existing graphs, Cayley graphs with better performance are in demand. Inspired by the work in [6], an interesting family of graphs, called *recursive cubes of rings*, were proposed as interconnection networks in [27]. A recursive cube of rings is not necessarily a Cayley graph, as shown in [31,15] by counterexamples to [27, Property 4]. Nevertheless, under a natural condition this graph is indeed a Cayley graph as we will see later. In [4] the vertex-disjoint paths problem for recursive cubes of rings was solved by using Hamiltonian circuit Latin squares, and in [27] the recursive construction of them was given. The diameter problem for recursive cubes of rings has attracted considerable attention: An upper bound was given in [27, Property 5] but shown to be incorrect in [31, Example 6]; and another upper bound was given in [31, Theorem 13] but it was unknown whether it gives the exact value of the diameter. A result in [15] on the diameter of a recursive cube of rings was also shown to be incorrect in [31].

1.1. Main results

The purpose of this paper is to conduct a comprehensive study of recursive cubes of rings. As mentioned above, a recursive cube of rings as defined in [27,31] is not necessarily a Cayley graph. We will give a necessary and sufficient condition for this graph to be a Cayley graph (see Theorem 2.7). We will see that, under this condition (given in (2)), a recursive cube of rings as in [31] can be equivalently defined as a Cayley graph on the semidirect product of an elementary abelian group by a cyclic group (see Definition 2.1). We believe that this definition is more convenient for studying various network properties of recursive cubes of rings. For example, from our definition it follows immediately that the cube-connected cycles [22] and cube-of-rings [6] are special recursive cubes of rings.

The above-mentioned condition (see (2)) will be assumed from Section 3 onwards. In Section 3, we give a method for finding a shortest path between any two vertices and a formula for the distance between them in a recursive cube of rings (see Theorems 3.2 and 3.3). In Section 4, we give an exact formula for the diameter of any recursive cube of rings (see Theorem 4.1). This result shows that the upper bound for the diameter given in [31] is not tight in general, though it is sharp in a special case. In Section 5, we give nearly matching lower and upper bounds on the Wiener index of a recursive cube of rings, expressed in terms of the total distance from a fixed vertex to all other vertices (see Theorems 5.2 and 5.4). These results will be used in Section 6 to obtain the vertex-forwarding index (see Theorem 6.1) and nearly matching lower and upper bounds for the edge-forwarding index (Theorem 6.6) of a recursive cube of rings. Another tool for obtaining the latter is the theory [25] of integral uniform flows in orbital-proportional graphs. In Section 7, we give nearly matching lower and upper bounds for the bisection width of a recursive cube of rings, which improve the existing upper bounds in [15,27,31].

Since the cube-connected cycles [22] and cube-of-rings [6] are special recursive cubes of rings, all results obtained in this paper are valid for these well known networks. In particular, we recover a couple of existing results for them in a few case, and obtain new results for them in the rest cases. All results in the paper are also valid for the network $RCR\text{-}II(d, r, n-d)$ [31] with $dr \equiv 0 \bmod n$ (see the discussion in Section 2.2).

Our study in this paper shows that recursive cubes of rings enjoy fixed degree, logarithmic diameter and relatively small forwarding indices in some cases, and flexible choice of order and other invariants when their defining parameters vary. Therefore, they are promising topologies for interconnection networks.

1.2. Terminology and notation

All graphs considered in the paper are undirected graphs without loops and multi-edges unless stated otherwise. Since any interconnection network can be modelled as a graph, we use the terms 'graph' and 'network' interchangeably.

A *path* of *length* n between two vertices u and v in a graph X is a sequence $u = u_0, e_1, u_1, e_2, \ldots, u_{n-1}, e_n, u_n = v$, where $u_0, u_1, \ldots, u_{n-1}, u_n$ are pairwise distinct vertices of X and $e_1, e_2, \ldots, e_n$ are pairwise distinct edges of X such that e_i is the edge joining u_{i-1} and u_i, $1 \le i \le n$. We may simply represent such a path by $u_0, u_1, \ldots, u_{n-1}, u_n$ or $e_1, e_2, \ldots, e_n$. A path between u and v with minimum length is called a *shortest path* between u and v. The *distance* between u and v in X, denoted by $\mathsf{dist}(u, v)$, is the length of a shortest path between them in X, and is ∞ if there is no path in X between u and v. The diameter of X is defined as $\mathsf{diam}(X) := \max_{u,v \in V(X)} \mathsf{dist}(u, v)$. The *Wiener index* of X is defined as $W(X) := \sum_{u,v \in V(X)} \mathsf{dist}(u, v)$, with the sum over all unordered pairs of vertices u, v of X. The Wiener index is important for chemical graph theory [3] but is difficult to compute in general. It is also used to estimate (or compute) the edge-forwarding index of a network (see [33,28,30]).

A permutation of $V(X)$ is called an *automorphism* of X if it preserves the adjacency and non-adjacency relations of X. The set of all automorphisms of X under the usual composition of permutations is a group, $\mathrm{Aut}(X)$, called the *automorphism group* of X. If $\mathrm{Aut}(X)$ is transitive on $V(X)$, namely any vertex can be mapped to any other vertex by an automorphism of X, then X is called *vertex-transitive*. The definition of an *edge-transitive* graph is understood similarly.

If X is vertex-transitive, then define the *total distance* $\mathsf{td}(X)$ of X to be the sum of the distances from any fixed vertex to all other vertices in X. It can be easily seen that, for a vertex-transitive graph X, the average distance of X is equal to $\mathsf{td}(X)/(|V(X)| - 1)$ and the Wiener index of X is given by $W(X) = |V(X)|\mathsf{td}(X)/2$.