Discrete Applied Mathematics 217 (2017) 663–672

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On weight choosabilities of graphs with bounded maximum average degree

Jakub Przybyło [a,*], André Raspaud [b], Mariusz Woźniak [a]

[a] AGH University of Science and Technology, al. A. Mickiewicza 30, 30-059 Krakow, Poland

[b] LaBRI, Universite de Bordeaux, 351, Cours de la Liberation, 33405 Talence Cedex, France

ARTICLE INFO

Article history:
Received 18 February 2015
Received in revised form 23 September 2016
Accepted 27 September 2016
Available online 20 October 2016

Keywords:
1–2–3 Conjecture
1–2-Conjecture
3-edge-weight choosability
2-total weight choosability
Combinatorial Nullstellensatz
Discharging method
Maximum average degree

ABSTRACT

The well known 1–2–3-Conjecture asserts that every connected graph G of order at least three can be edge-coloured with integers 1, 2, 3 so that the sums of colours met by adjacent vertices are distinct in G. The same is believed to hold if instead of edge colourings we consider total colourings assigning 1 or 2 to every vertex and edge of a given graph—this time the colour of every vertex is counted in its corresponding sum. We consider list extensions of the both concepts, where every edge (and vertex) is assigned a set of k positive integers, i.e., its potential colours, and regardless of this list assignment we wish to be able to construct a colouring from these lists so that the adjacent vertices are distinguished by their corresponding sums. We prove that if the maximum average degree of the graph G is smaller than $\frac{5}{2}$, then lists of length $k = 3$ are sufficient for that goal in case of edge colourings (if G has no isolated edges), while already $k = 2$ suffices in the total case. In fact the second of these statements remains true with arbitrary real colours admitted instead of positive integers, and the first one—for positive reals. The proofs of these facts are based on the discharging method combined with the algebraic approach of Alon known as *Combinatorial Nullstellensatz*.

1. Introduction

It is a well known fact that there are no *irregular graphs*, understood as simple graphs with pairwise distinct vertex degrees (cf. [6] for possible alternative definitions of irregular graphs), except for the trivial 1-vertex case. A special invariant was thus introduced in [7] aiming at capturing a level of an irregularity of a graph. Suppose that given a graph $G = (V, E)$ we wish to construct a multigraph with pairwise distinct vertex degrees of it by multiplying some of its edges. The least k so that we are able to achieve such goal using at most k copies of every edge is denoted by $s(G)$ and referred to as the *irregularity strength* of G. Alternatively one may consider (not necessarily proper) edge colourings $c : E \to \{1, 2, \ldots, k\}$ with $\sum_{e\ni u} c(e) \neq \sum_{e\ni v} c(e)$ for every pair of distinct vertices $u, v \in V$. Then the least k which permits defining a colouring c with this feature equals $s(G)$. Numerous papers have been devoted to study on this graph invariant since the middle 80s, see e.g. [9,10,14,16–20].

On the other hand, there are many *locally irregular graphs*, i.e., those whose only *adjacent* vertices are required to have distinct degrees, see [5,21] for studies devoted to these. Analogously as above one might thus also measure how far a given graph $G = (V, E)$ is from being locally irregular. That is, how large k is needed in order to define an edge colouring

* Corresponding author. Fax: +48 12 617 31 65.
E-mail addresses: jakubprz@agh.edu.pl (J. Przybyło), raspaud@labri.fr (A. Raspaud), mwozniak@agh.edu.pl (M. Woźniak).

http://dx.doi.org/10.1016/j.dam.2016.09.037
0166-218X/

$c : E \to \{1, 2, \ldots, k\}$ with $\sum_{e\ni u} c(e) \neq \sum_{e\ni v} c(e)$ for every pair of adjacent vertices u, v of G. It is believed that already $k = 3$ is sufficient for all graphs containing no isolated edges. This suspicion is commonly referred to as the 1–2–3-Conjecture, see [15], and still widely studied, as thus far it is only known that $k = 5$ suffices, see [13], and [1,2,8] for other interesting results. The last breakthrough concerning this graph invariant was obtained due to development of an algorithm designed to attack a total analogue of the 1–2–3-Conjecture. This investigates how large k is needed in order to define a total colouring $c : V \cup E \to \{1, 2, \ldots, k\}$ with $c(u) + \sum_{e\ni u} c(e) \neq c(v) + \sum_{e\ni v} c(e)$ for every pair of adjacent vertices u, v in a graph $G = (V, E)$. The 1–2-Conjecture, see [22], asserts that $k = 2$ suffices for every graph. This is known to hold with additionally colour 3 admitted for the edges as proved by Kalkowski [12] (thus the conjecture itself remains open). The both conjectures are also known to hold e.g. for graphs with maximum average degree less than $\frac{8}{3}$, as proved by Cranston, Jahanbekam and West in [8].

Given any colouring $c : E \to \mathbb{R}$ (or $c : V \cup E \to \mathbb{R}$), whose values are frequently also called *weights* of the edges (and vertices) in the literature, we shall denote by

$$d_c(v) := \sum_{u\in N(v)} c(uv) \qquad \left(\text{or } d_c(v) := c(v) + \sum_{u\in N(v)} c(uv),\ \text{resp.}\right)$$

the *weighted degree* of $v \in V$.

Now suppose we investigate a natural extension of the 1–2–3-Conjecture (the local correspondent of the irregularity strength), and instead of admitting colours $1, 2, \ldots, k$ for every edge of G, we are forced to use colours from (possibly distinct) k-element subsets of the set $\{1, 2, 3, \ldots\}$ prescribed to the edges. In this paper we develop this extension and admit also positive *real* weights (i.e., weights from $\mathbb{R}_+$) in such lists associated with the edges. We shall say that the graph $G = (V, E)$ is *k-positive-edge weight choosable* if for every list assignment $L : E \to 2^{\mathbb{R}_+}$ with $|L(e)| = k$ for $e \in E$, there exists an edge colouring $c : E \to \mathbb{R}_+$ with $c(e) \in L(e)$ for $e \in E$ (i.e., a *colouring from the given lists*) such that $d_c(u) \neq d_c(v)$ for every $uv \in E$. A generalization of this concept, admitting lists of arbitrary real numbers was priory introduced under the name of *k-weight choosability* in [4], where Bartnicki et al. conjectured that every graph without an isolated edge is 3-weight choosable. This generalization of the 1–2–3-Conjecture was proved to hold e.g. for complete graphs, complete bipartite graphs and trees by these authors. Later total analogues of the above were introduced by Przybyło and Woźniak in [23], and independently by Wong and Zhu in [26]. We say that the graph $G = (V, E)$ is *k-total weight choosable* if for every list assignment $L : V \cup E \to 2^{\mathbb{R}}$ with $|L(v)|, |L(e)| = k$ for $v \in V, e \in E$, there exists a total colouring $c : V \cup E \to \mathbb{R}$ with $c(v) \in L(v), c(e) \in L(e)$ for $v \in V, e \in E$ such that $d_c(u) \neq d_c(v)$ for every $uv \in E$. It was conjectured in [23,26] that every graph is 2-total weight choosable. This has been confirmed for several classes of graph including complete graphs, trees, unicyclic graphs, wheels, generalized theta graphs, complete bipartite graphs, and a special class of complete multipartite graphs, see [23,24,26]. Moreover, in [27] Wong and Zhu developed a beautiful and clever argument, generalizing the result of Kalkowski [12], i.e., proving that lists of length 2 and 3 for the vertices and edges, resp., are always sufficient to construct a desired total colouring. In fact Wong and Zhu [26] introduced even more general concept in the total case. We say that a graph $G = (V, E)$ is (k, k')-choosable if for every list assignment $L : V \cup E \to 2^{\mathbb{R}}$ with $|L(v)| = k, |L(e)| = k'$ for $v \in V, e \in E$, there exists a total colouring $c : V \cup E \to \mathbb{R}$ with $c(v) \in L(v), c(e) \in L(e)$ for $v \in V, e \in E$ such that $d_c(u) \neq d_c(v)$ for every $uv \in E$. They also conjectured that every graph without isolated edges is $(1, 3)$-choosable. This is still open in general. We shall refer to it as the $(1, 3)$-choosable Conjecture.

2. Results and tools

The *maximum average degree* is a well known graph invariant, which is a conventional measure of sparseness of an arbitrary graph defined as

$$\mathrm{mad}(G) = \max\left\{\frac{2|E(H)|}{|V(H)|},\ H \subseteq G\right\}.$$

For more details on this invariant see [11], where properties of the maximum average degree are exhibited and where it is proved that the maximum average degree may be computed by a polynomial algorithm.

The main results of this paper are the two following theorems.

Theorem 1. *Every graph G without an isolated edge and with $\mathrm{mad}(G) < \frac{5}{2}$ is 3-positive-edge weight choosable.*

Theorem 2. *Every graph G with $\mathrm{mad}(G) < \frac{5}{2}$ is 2-total weight choosable.*

We thus show that graphs with $\mathrm{mad}(G) < \frac{5}{2}$ satisfy the thesis of the list correspondent of the 1–2-Conjecture from [23,26], and the same is true in a special subcase of the conjecture from [4] and the $(1, 3)$-choosable Conjecture, where all lists associated with the edges contain only positive numbers, while vertices may be regarded to have the list $\{0\}$ assigned. It is worth mentioning that a stronger result than Theorem 2, namely that every 2-degenerate graph is 2-total weight choosable, was recently independently obtained in yet unpublished paper [25] together with a few other interesting results.