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A new generalization of kernels in digraphs

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ABSTRACT

Let D be a digraph and k, l two positive integers. A subset N is a (k, l)-out-kernel of D if and only if N is a k-independent and a l-out-dominating set of D (that is $\Delta^+(N) < k$ and $\forall x \in V \setminus N$, $\left|N_D^+(x) \cap N\right| \geq l$). A digraph such that every induced subdigraph has a (k, l)-out-kernel is called (k, l)-out-kernel perfect.

A k-out-kernel is a (k, k)-out-kernel. Under this definition a kernel is a 1-out-kernel or a $(1, 1)$-out-kernel.

Since an $(n-1)/2$-regular digraph with an odd order does not have an $(n-1)/2$-out-kernel, the natural question is: which digraphs have a (k, l)-out-kernel or a k-out-kernel?

In this paper we investigate the problem of the existence of a (k, l)-out-kernel and a k-out-kernel in digraphs, and generalize some classical results on kernels in digraphs.

1. Introduction

A digraph D is an ordered pair $(V(D), A(D))$, where $V(D)$ is a finite set and $A(D)$ is a set of ordered pairs xy with $x, y \in V(D)$ (i.e., $A(D)$ is a binary relation on $V(D)$). The elements of $V(D)$ are vertices and the elements of $A(D)$ are arcs. In our definitions of a digraph, we do not allow multiple arcs, i.e., arcs joining two vertices x and y in the same direction, and we do not allow loops, i.e., arcs joining a vertex to itself. We will use $D = (V, A)$ or simply D to denote a digraph. Let $D = (V, A)$ be a digraph, an arc $uv \in A$ is called asymmetrical (resp. symmetrical) if $vu \notin A$ (resp. $vu \in A$). The asymmetrical part of D, Asym(D) is the subdigraph of D such that $V(\mathrm{Asym}(D)) = V(D)$ and $A(\mathrm{Asym}(D)) = \{uv \in A \mid uv \text{ is asymmetrical}\}$. A digraph D is called symmetric if uv is an arc of D whenever vu is, i.e., $A(\mathrm{Asym}(D)) = \varnothing$. A digraph D is called transitive if uw is an arc of D whenever uv and vw are, where u, v and w are different vertices. We call $D = (V, A)$ an oriented graph if the relation A is antisymmetric, i.e., $A(\mathrm{Asym}(D)) = A(D)$. A digraph $D = (V, A)$ is called semi-complete if for every two distinct vertices $u, v \in V$, $uv \in A$ or $vu \in A$. A tournament is a complete asymmetrical digraph.

For every undirected simple graph $G = (V, E)$, we call orientation of G any directed graph D with the same vertex-set as G and such that two vertices are adjacent in G if and only if there exists at least one arc between them in D (notice that an edge in G may correspond to a pair of symmetric arcs in D).

We use $UG(D)$ to denote the underlying graph of D, i.e., the simple graph with vertex set V and (u, v) is an edge of $UG(D)$ if and only if uv or vu is an arc of D. Throughout this paper, any digraph D is to be viewed as an orientation of its underlying graph.

If xy is an arc of a digraph D, then we say that y absorbs x and x dominates y and y is an out-neighbour of x, and x is an in-neighbour of y. The outset of x, denoted by $N_D^+(x)$, consists of all out-neighbours of x, and the closed outset of x, denoted by $N_D^+[x]$ is just $N_D^+(x) \cup \{x\}$. The inset of x, denoted by $N_D^-(x)$, consists of all in-neighbours of x, and the closed inset of x, denoted by $N_D^-[x]$ is just $N_D^-(x) \cup \{x\}$. The number of vertices in $N_D^+(x)$, denoted by $d_D^+(x) = \left|N_D^+(x)\right|$ is the outdegree of x, and the

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number of vertices in $N_D^-(x)$, denoted by $d_D^-(x) = \left|N_D^-(x)\right|$, is the indegree of x. We use $\Delta^+(D)$ and $\Delta^-(D)$ to denote the maximum outdegree and the maximum indegree in D, respectively and $\delta^+(D)$ and $\delta^-(D)$ to denote the minimum outdegree and the minimum indegree of D, respectively. A digraph D is k-regular if all vertices of D have indegree and outdegree equal to k.

Let $D = (V, A)$ and $D' = (V', A')$ be two digraphs, we call D' a subdigraph of D if $V' \subseteq V$ and $A' \subseteq A$. If in addition $A' = \{xy \in A \mid x, y \in V'\}$ then we call D' an induced subdigraph of D. For each $S \subseteq V(D)$, the subdigraph of D induced by S, denoted by $D[S]$, is the unique induced subdigraph of D with vertex set S. The outset of S (resp. the closed outset of S) is $N_D^+(S) = \bigcup_{u\in S} N_D^+(u)$ (resp. $N_D^+[S] = N_D^+(S) \cup S$) and the inset of S (resp. the closed outset of S) is $N_D^-(S) = \bigcup_{u\in S} N_D^-(u)$ (resp. $N_D^-[S] = N_D^-(S) \cup S$). We use $\Delta^+(S)$, $\Delta^-(S)$, $\delta^+(S)$ and $\delta^-(S)$ for $\Delta^+(D[S])$, $\Delta^-(D[S])$, $\delta^+(D[S])$ and $\delta^-(D[S])$, respectively. The maximum degree $(\Delta^+(S) + \Delta^-(S))$ (resp. the minimum degree $\left(\delta^+(S) + \delta^-(S)\right)$) of $D[S]$ is denoted by $\Delta(S)$ (resp. $\delta(S)$). The number of vertices in $N_S^+(x) = N_D^+(x) \cap S$, denoted by $d_S^+(x) = \left|N_S^+(x)\right|$ is the outdegree of x in S, and the number of vertices in $N_S^-(x) = N_D^-(x) \cap S$, denoted by $d_S^-(x) = \left|N_S^-(x)\right|$, is the indegree of x in S.

A set $S \subseteq V(D)$ is independent if $A(D[S]) = \varnothing$, i.e., $\Delta^+(S) = 0$. A kernel N of D is an independent set of vertices such that every vertex $z \in V(D) \setminus N$ is absorbed by at least one vertex of N. A digraph D is called a kernel-perfect digraph if every induced subdigraph of D has a kernel.

A directed path P of length k is a digraph with the vertex set $\{x_0, x_1, \ldots, x_k\}$ and the arc set $\{x_0x_1, x_1x_2, \ldots, x_{k-1}x_k\}$, such that all vertices and arcs shown are distinct. We will call such a directed path an x_0x_k-path and will denote it by $x_0x_1 \ldots x_{k-1}x_k$. A directed cycle C of length k is a digraph with vertex set $\{x_1, \ldots, x_k\}$ and the arc set $\{x_1x_2, x_2x_3, \ldots, x_{k-1}x_k, x_kx_1\}$, such that all vertices and arcs shown are distinct. We will denote such a directed cycle by $x_1 \ldots x_{k-1}x_kx_1$. A directed cycle is an odd directed cycle if k is odd and is an even directed cycle if k is even.

By $d_D(x, y)$ we denote the length of the shortest path from x to y in a digraph D. If there does not exist a path from x to y in D then we put $d_D(x, y) = \infty$.

A digraph D is strong if for any two vertices x and y there is a directed xy-path and there is directed yx-path. A strong component of the digraph D is a maximal strong subdigraph. The strong component digraph or the condensation $SC(D)$ of a digraph D is obtained by contracting each strong component C to a single vertex $v(C)$, and there is an arc from a vertex $v(C)$ to a vertex $v(C')$ in $SC(D)$ if there exists at least one arc from a vertex in C to a vertex in C'. Clearly $SC(D)$ is an acyclic digraph (a digraph without directed cycles). A terminal component of D is a strong component C_T of D such that $d^+_{SC(D)}(v(C_T)) = 0$, an initial component of D is a strong component C_I of D such that $d^-_{SC(D)}(v(C_I)) = 0$. All terms not defined here may be found in the book of C. Berge [2].

Domination and other related concepts in undirected graphs are well studied, the respective analogs on digraphs have not received much attention. The pioneering work in digraphs in this area can be ascribed to Berge, Harary, König, Grundy and Richardson, amongst others.

The concept of kernel was introduced by Von Neumann and Morgenstern [31], as an abstract generalization of their concept of solution for cooperative games. They also proved that any finite acyclic digraph has an unique kernel. So the main problem is: Which structural properties of a digraph imply the existence of a kernel? The concept of a kernel in digraphs has attracted much attention because some problems may be modelled by use of digraph kernels. This concept has found applications for instance in cooperative n-person games, in Nim-type games [2], in logic [1], etc. The existence of a kernel is a difficult problem: Chvatal [7] showed that deciding if a digraph possesses a kernel is an NP-complete problem, Fraenkel [15] showed that it remains NP-complete for planar digraphs with small degrees. In particular, Hell and Hernandez Cruz [22] prove that the problem of determining whether a digraph has a 3-kernel is NP-complete and as consequence the kernel problem is proved to be NP-complete for 3-colourable digraphs. Many sufficient conditions for the existence of a kernel are known. Such conditions are usually hereditary and so they also imply the existence of a kernel for every induced subdigraph. A digraph such that every induced subdigraph has a kernel is called kernel perfect. Some classical results on the existence of kernels in digraphs are: a symmetric digraph is kernel perfect, a transitive digraph is kernel perfect, and all kernels have the same cardinality (König), a digraph without cycles is kernel perfect, and its kernel is unique (Von Neumann), a digraph without cycles of odd length is kernel perfect (Richardson). Indeed, many extensions of Richardson's Theorem have been found. The problem of the existence of a kernel in a given digraph has been studied by several authors in particular by Richardson [28,27], Duchet and Meyniel [10,11], Duchet [8,9], Blidia [4], Galeana-Sánchez and Neumann-Lara [19].

In the literature we know two generalizations of concept of kernel in digraphs.

The first one is the concept of kernels by monochromatic paths. We call the digraph D an m-coloured digraph if the arcs of D are coloured with m colours. A subdigraph H of D is called monochromatic if all of its arcs are coloured with the same colour. A set $N \subseteq V(D)$ is called a kernel by monochromatic paths of D, if for every pair of distinct vertices $u, v \in N$ there is no monochromatic directed path between them (which means that N is independent by monochromatic paths) and for every vertex $x \in V(D) \setminus N$ there is a monochromatic path from x to a vertex $y \in N$ (which means that N is absorbent by monochromatic paths). An m-coloured digraph is called kernel-perfect by monochromatic paths if every induced subdigraph of D has a kernel by monochromatic paths. It is easy to see that every 1-coloured digraph has a kernel by monochromatic paths, because every transitive digraph has a kernel. The study of existence of kernels by monochromatic paths in m-coloured digraphs starts with the theorem of Sands, Sauer and Woodrow [29], which asserts that every 2-coloured digraph has a