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Note

Graph connectivity and universal rigidity of bar frameworks

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ABSTRACT

Let G be a graph on n nodes. In this note, we prove that if G is $(r+1)$-vertex connected, $1 \le r \le n-2$, then there exists a configuration p in general position in $\mathbb{R}^r$ such that the bar framework (G, p) is universally rigid. The proof is constructive, and is based on a theorem by Lovász *et al* concerning orthogonal representations and connectivity of graphs Lovász et al. (0000, 2000).

1. Introduction

Let $G = (V, E)$ be a graph on n nodes. G is said to be *k-vertex connected*, or simply *k-connected*, if $n = k+1$ and G is the complete graph, or if $n \ge k+2$ and there does not exist a set of $(k-1)$ nodes whose deletion disconnects G. A *bar framework* in $\mathbb{R}^r$ is a simple incomplete connected graph G whose nodes are points $p^1, \ldots, p^n$ in $\mathbb{R}^r$; and whose edges are line segments, each joining a pair of these points. The points $p^1, \ldots, p^n$ will be denoted collectively by p, and the bar framework will be denoted by (G, p). Also, we will refer to p as the *configuration* of the bar framework. A configuration p (or a framework (G, p)) is *r-dimensional* if the points $p^1, \ldots, p^n$ affinely span $\mathbb{R}^r$. Moreover, a configuration p (or a framework (G, p)) is in general position in $\mathbb{R}^r$ if every $r+1$ points in configuration p are affinely independent.

An r'-dimensional bar framework (G, p') is *equivalent* to an r-dimensional bar framework (G, p) if:

$$\|p'^i - p'^j\|^2 = \|p^i - p^j\|^2 \quad \text{for each } \{i, j\} \in E(G), \tag{1}$$

where $\|.\|$ denotes the Euclidean norm and $E(G)$ denotes the edge set of graph G. On the other hand, two r-dimensional bar frameworks (G, p) and (G, p') are *congruent* if:

$$\|p'^i - p'^j\|^2 = \|p^i - p^j\|^2 \quad \text{for all } i, j = 1, \ldots, n. \tag{2}$$

An r-dimensional bar framework (G, p) is said to be *universally rigid* if there does not exist an r'-dimensional bar framework (G, p'), where r' is a positive integer $\le n-1$, such that (G, p') is equivalent but not congruent to (G, p).

An immediate necessary condition for an r-dimensional bar framework (G, p) on n nodes $(r \le n-2)$ in general position in $\mathbb{R}^r$ to be universally rigid is that graph G should be $(r+1)$-connected [11]. For suppose that G is not $(r+1)$-connected. Then there exists a set of r nodes, say X, whose removal disconnects G. Let $V(G) = V_1 \cup X \cup V_2$ be a partition of the nodes of G, where V_1 and V_2 are non-empty, such that there are no edges joining nodes in V_1 to nodes in V_2. The points $\{p^i : i \in X\}$ lie in a hyperplane H in $\mathbb{R}^r$, and the points $\{p^i : i \in V_1 \cup V_2\}$ do not lie in H since p is in general position in $\mathbb{R}^r$. For all nodes

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$i \in V_2$, let q^i be the reflection of p^i with respect to H and let $p' = \{p^i : i \in (V_1 \cup X)\} \cup \{q^i : i \in V_2\}$. Thus (G, p') is an r-dimensional bar framework that is equivalent but not congruent to (G, p), and hence (G, p) is not universally rigid. This raises the question of whether the assumption of $(r+1)$-connectivity of graph G alone is sufficient for the existence of some r-dimensional configuration p in general position in $\mathbb{R}^r$ such that the bar framework (G, p) is universally rigid. The following theorem, which is our main result, is an affirmative answer to this question.

Theorem 1.1. *Let G be a graph on n nodes and assume that G is $(r + 1)$-vertex connected, where $1 \leq r \leq n - 2$. Then there exists an r-dimensional bar framework (G, p) in general position in $\mathbb{R}^r$ such that (G, p) is universally rigid.*

The proof of Theorem 1.1, which is given in Section 3, is constructive and is based on a theorem by Lovász et al. [13,14] concerning orthogonal representations and connectivity of graphs. There is a vast literature on orthogonal representations of graphs [12]. But section 6 of [15] is the most relevant to this note.

Consider the complete bipartite graph $K_{3,3}$ with vertex partitions {1, 3, 5} and {2, 4, 6}. Obviously, $K_{3,3}$ is 3-connected. Hence, it should come as no surprise that there exists a configuration p in general position in $\mathbb{R}^2$ such that the framework $(K_{3,3}, p)$ is universally rigid. For assume that the vertices of $(K_{3,3}, p)$ lie on a conic in the plane; i.e., $(p^i)^T A p^i + b^T p^i + \sigma = 0$ for all $i = 1, \ldots, 6$, where A is a non-zero 2×2 symmetric matrix, $b \in \mathbb{R}^2$ and σ is a scalar. Then $(K_{3,3}, p)$ has a non-zero stress matrix Ω [5, Theorem 14] (see the definition below). Moreover, if in addition, the points $p^1, \ldots, p^6$ are the ordered corners of a convex polygon, then it follows from [6, Theorem 5] that Ω is positive semidefinite and of rank 3 and hence, $(K_{3,3}, p)$ is universally rigid (see Theorem 2.1).

2. Preliminaries

This section presents the necessary mathematical background. The first subsection reviews basic definitions and results on stress and Gale matrices, and their role in the problem of universal rigidity. The second subsection focuses on vertex connectivity and orthogonal representations of graphs.

2.1. Stress and Gale matrices

Stress matrices play a key role in the study of universal rigidity. An *equilibrium stress* (or simply a *stress*) of a bar framework (G, p) is a real-valued function ω on $E(G)$ such that:

$$\sum_{j:\{i,j\}\in E(G)} \omega_{ij}(p^i - p^j) = \mathbf{0} \quad \text{for each } i = 1, \ldots, n. \tag{3}$$

Here we use the bold zero "$\mathbf{0}$" to denote the zero vector or the zero matrix of appropriate dimensions. Let $E(\overline{G})$ denote the edge set of graph $\overline{G}$, the complement graph of G. i.e.,

$$E(\overline{G}) = \{\{i, j\} : i \neq j, \{i, j\} \notin E(G)\},$$

and let $\omega = (\omega_{ij})$ be a stress of (G, p). Then the $n \times n$ symmetric matrix Ω where

$$\Omega_{ij} = \begin{cases} -\omega_{ij} & \text{if } \{i,j\} \in E(G), \\ 0 & \text{if } \{i,j\} \in E(\overline{G}), \\ \displaystyle\sum_{k:\{i,k\}\in E(G)} \omega_{ik} & \text{if } i = j, \end{cases} \tag{4}$$

is called the *stress matrix associated with* ω, or a *stress matrix* of (G, p). Sufficient and necessary conditions, in terms of stress matrices, for universal rigidity of bar frameworks are discussed in [1,7,6,2,9]. The first sufficient condition for universal rigidity under the assumption that configuration p is in general position was given in [4].

Theorem 2.1 (*Alfakih and Ye [4]*)**.** *Let (G, p) be an r-dimensional bar framework on n nodes in $\mathbb{R}^r$, for some $r \leq n - 2$. If the following two conditions hold:*

1. *There exists a positive semidefinite stress matrix Ω of (G, p) of rank $n - r - 1$,*
2. *The configuration p is in general position.*

Then (G, p) is universally rigid.

Theorem 2.1 was generalized and strengthened in [3], but it will suffice for the purposes of this note.

Stress matrices are intimately related to Gale matrices and Gale transform [8,10]. This relation is a crucial step in connecting stress matrices to orthogonal representations of graphs. Given an r-dimensional bar framework (G, p) in $\mathbb{R}^r$, let

$$P := \begin{bmatrix} (p^1)^T \\ \vdots \\ (p^n)^T \end{bmatrix}. \tag{5}$$