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Characterization of the allowed patterns of signed shifts

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ABSTRACT

The allowed patterns of a map are those permutations in the same relative order as the initial segments of orbits realized by the map. In this paper, we characterize and provide enumerative bounds for the allowed patterns of signed shifts, a family of maps on infinite words.

1. Introduction

The study of allowed and forbidden patterns associated to a map bridges the fields of enumerative combinatorics and dynamical systems. As stated below, there are various consequences to characterizing and enumerating allowed and forbidden patterns, and in addition, studying these ideas has also led to purely combinatorial results [6,8,9] in both enumerative and algebraic combinatorics. These ideas have also led to and contributed to the study of so-called infinite permutations (see, for example, [13]).

Various results have been proven about the allowed patterns of a piecewise monotone map on the unit interval. For example, if f is such a map, then the size of $\mathcal{A}_n(f)$ grows at most exponentially [7], while the number of permutations grows super-exponentially and thus f will have forbidden patterns. One application of this is that one may distinguish a random time series from a deterministic one [4,5], since a random time series will eventually contain all patterns, while most patterns are forbidden in a deterministic time series. In addition, the size of $|\mathcal{A}_n(f)|$ for a given f is known to be directly related to the topological entropy of f, a value which measures the complexity of the map [7].

For these reasons, characterizing and enumerating the allowed patterns of a given map f presents an interesting problem. Previously, the question of characterizing and enumerating allowed patterns has been answered for the well-known left shift on binary words in [15], on words on a general alphabet (called the k-shift) in [8], for β-shifts in [10], for negative β-shifts [12], and has been partially addressed for logistic maps [11]. In this paper, we provide a characterization of the allowed patterns for a family of maps called the signed shifts, which generalize the k-shift and the well-known tent map, as well as bounds on the enumeration of these patterns. Though we do not approach the question of characterizing the forbidden (or minimal forbidden) patterns of signed shifts, this could be an interesting question for future study.

The problem of characterizing the allowed patterns which are realized by the signed shifts have been studied in several papers including [1–3,8]. In [1], the author presents a partial characterization of these permutations. In this paper, we show that the conditions presented in [1] are not sufficient for the permutations to be allowed and present a complete characterization of the allowed patterns of signed shifts. In Section 2, we present some necessary background and some of

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the ideas involved in the proof. In Section 3, we state counterexamples to the sufficiency of the characterization in [1] and describe the corrected characterization in Theorem 3.10. In Section 4, we provide the proof of the characterization, which is quite technical. In Section 5, we provide bounds on the number of allowed patterns of size n for each signed shift and address a few special cases.

2. Background

2.1. Permutations

We denote by $\mathcal{S}_n$ the set of permutations of $[n] = \{1, 2, \ldots, n\}$ and write permutations in one-line notation as $\pi = \pi_1\pi_2\ldots\pi_n \in \mathcal{S}_n$. Occasionally, we will write a permutation in cycle notation as a product of disjoint cycles. A *cyclic permutation*, or *cycle*, is a permutation $\pi \in \mathcal{S}_n$ which is composed of a single n-cycle. We denote the set of cyclic permutations of length n by $\mathcal{C}_n$. For example, the permutation $\pi = 37512864 = (13527684)$ is a cyclic permutation in $\mathcal{C}_8$ written in both its one-line notation and cycle notation.

It will be useful to define the map

$$\begin{aligned} \mathcal{S}_n &\to \mathcal{C}_n \\ \pi &\mapsto \hat{\pi}, \end{aligned}$$

where if $\pi = \pi_1\pi_2\ldots\pi_n$ in one-line notation, then $\hat{\pi} = (\pi_1, \pi_2, \ldots, \pi_n)$ in cycle notation, that is, $\hat{\pi}$ is the cyclic permutation that sends π_1 to π_2, π_2 to π_3, and so on. Writing $\hat{\pi} = \hat{\pi}_1\hat{\pi}_2\ldots\hat{\pi}_n$ in one-line notation, we have that $\hat{\pi}_{\pi_i} = \pi_{i+1}$ for $1 \le i \le n$, with the convention that $\pi_{n+1} := \pi_1$. Notice that this map sends all cyclic rotations of π to the same cyclic permutation $\hat{\pi}$. For example, if $\pi = 17234856$, then $\hat{\pi} = (17234856) = 73486125$. The map $\pi \mapsto \hat{\pi}$ also appears in [8,6].

As defined in [6], we say that a permutation $\tau \in \mathcal{S}_n$ is an element of the σ-*class*, denoted by $\mathcal{S}^\sigma$, if that there is some sequence, $0 = e_0 \le \cdots \le e_k = n$, which we call a σ-*segmentation*, so that $\tau_{e_t+1}\cdots\tau_{e_{t+1}}$ is increasing if $\sigma_t = +$ and decreasing if $\sigma_t = -$.

Example 2.1. The permutation $\tau = 3586124$ is in $\mathcal{S}^\sigma$ for $\sigma = + - +$ since it has a σ-segmentation $0 \le 3 \le 6 \le 8$. Indeed, the segment 347 is increasing, 861 is decreasing, and 24 is increasing. Notice that this is not unique! For example, $0 \le 4 \le 6 \le 8$ is also a σ-segmentation of τ.

In Section 3, we will define similar concepts for $*$-*permutations*, which are typical permutations where one element has been replaced with a $*$.

2.2. Allowed patterns

Let X be a linearly ordered set and $x_1, x_2, \ldots, x_n \in X$ be distinct. Then we can define the *reduction operation* by

$$\rho(x_1, x_2, \ldots, x_n) = \pi$$

where $\pi = \pi_1\pi_2\ldots\pi_n$ is the permutation of $[n]$, written in one-line notation, whose entries are in the same relative order as the n entries in the input. For example, $\rho(3.3, 3.7, 9, 6, 0.2) = 23541$.

Consider a map $f : X \to X$. Iterating this map f at a point $x \in X$ returns a sequence of elements from X called the *orbit* of x with respect to f:

$$x, f(x), f^2(x), \ldots.$$

If there are no repetitions among the first n elements of the orbit, then we define the *pattern* of x with respect to f of length n to be

$$\mathrm{Pat}(x, f, n) = \rho(x, f(x), f^2(x), \ldots, f^{n-1}(x)).$$

If $f^i(x) = f^j(x)$ for some $0 \le i < j < n$, then $\mathrm{Pat}(x, f, n)$ is not defined. The set of *allowed patterns* of f is the set of permutations which are realized by f in this way:

$$\mathcal{A}(f) = \{\mathrm{Pat}(x, f, n) : n \ge 0, x \in X\}.$$

We denote by $\mathcal{A}_n(f)$ the allowed patterns of length n. Permutations which are not allowed patterns are called the *forbidden patterns* of f.

Example 2.2. Consider the logistic map L on the unit interval defined by $L(x) = 4x(1-x)$. Then the pattern at $x = 0.3$ of length 3 with respect to L is the permutation 132 since the first 3 elements of the orbit 0.3, 0.84, 0.5376 are in the same relative order as 132. The allowed patterns of L of length 3 are $\mathcal{A}_3(f) = \{123, 132, 213, 231, 312\}$ and the forbidden patterns of L of length 3 are $\mathcal{S}_3 \setminus \mathcal{A}_3(L) = \{321\}$. Indeed, there is no $x \in [0, 1]$ so that $x, L(x), L(L(x))$ is in decreasing order.

We say that two maps $f : X \to X$ and $g : Y \to Y$ are *order-isomorphic* if there is an order-preserving bijection $h : X \to Y$ so that $h \circ f = g \circ h$. If f and g are order-isomorphic, then they have the same allowed patterns. Since h is order preserving, $\mathrm{Pat}(x, f, n) = \mathrm{Pat}(h(x), g, n)$. Indeed, $g \circ h(x) < g \circ h(y)$ if and only if $h \circ f(x) < h \circ f(y)$ which occurs if and only if $f(x) < f(y)$.

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