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Radio number of trees

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ABSTRACT

A radio labeling of a graph G is a mapping $f : V(G) \to \{0, 1, 2, \ldots\}$ such that $|f(u)-f(v)| \geq \mathrm{diam}(G) + 1 - d(u, v)$ for every pair of distinct vertices u, v of G, where $\mathrm{diam}(G)$ is the diameter of G and $d(u, v)$ the distance between u and v in G. The radio number of G is the smallest integer k such that G has a radio labeling f with $\max\{f(v) : v \in V(G)\} = k$. We give a necessary and sufficient condition for a lower bound on the radio number of trees to be achieved, two other sufficient conditions for the same bound to be achieved by a tree, and an upper bound on the radio number of trees. Using these, we determine the radio number for three families of trees.

1. Introduction

In a graph model for the channel assignment problem, the transmitters are represented by the vertices of a graph; two vertices are adjacent or at distance two apart in the graph if the corresponding transmitters are *very close* or *close* to each other. Motivated by this problem Griggs and Yeh [9] introduced the following distance-two labeling problem: An *$L(2, 1)$-labeling* of a graph $G = (V(G), E(G))$ is a function f from the vertex set $V(G)$ to the set of nonnegative integers such that $|f(u) - f(v)| \geq 2$ if $d(u, v) = 1$ and $|f(u) - f(v)| \geq 1$ if $d(u, v) = 2$, where $d(u, v)$ is the distance between u and v in G. The *span* of f is defined as $\max\{f(u) - f(v) : u, v \in V(G)\}$, and the minimum span over all $L(2, 1)$-labelings of G is called the *λ-number* of G, denoted by $\lambda(G)$. The $L(2, 1)$-labeling and other distance-two labeling problems have been studied by many researchers in the past two decades; see [5,22].

It has been observed that interference among transmitters may go beyond two levels. Motivated by the channel assignment problem for FM radio stations, Chartrand et al. [6] introduced the following radio labeling problem. Denote by $\mathrm{diam}(G)$ the *diameter* of G, that is, the maximum distance among all pairs of vertices in G.

Definition 1.1. A *radio labeling* of a graph G is a mapping $f : V(G) \to \{0, 1, 2, \ldots\}$ such that for every pair of distinct vertices u, v of G,

$$d(u, v) + |f(u) - f(v)| \geq \mathrm{diam}(G) + 1.$$

The integer $f(u)$ is called the *label* of u under f, and the *span* of f is defined as $\mathrm{span}(f) = \max\{|f(u) - f(v)| : u, v \in V(G)\}$. The *radio number* of G is defined as

$$\mathrm{rn}(G) := \min_{f} \mathrm{span}(f)$$

with minimum over all radio labelings f of G. A radio labeling f of G is *optimal* if $\mathrm{span}(f) = \mathrm{rn}(G)$.

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Observe that any radio labeling should assign different labels to distinct vertices. Note also that any optimal radio labeling must assign 0 to some vertex. In the case when $\mathrm{diam}(G) = 2$ we have $\mathrm{rn}(G) = \lambda(G)$.

Determining the radio number of a graph is an interesting but challenging problem. So far the radio number is known only for a handful families of graphs (see [8] for a survey). Chartrand et al. [6,7,23] studied the radio labeling problem for paths and cycles, and this was continued by Liu and Zhu [17] who gave the exact value of the radio number for paths and cycles. We emphasize that even in these innocent-looking cases it was challenging to determine the radio number. In [15,16], Liu and Xie discussed the radio number for the square of paths and cycles. In [18–20], Vaidya and Bantva studied the radio number for the total graph of paths, the strong product of P_2 with P_n and linear cacti. In [2], Benson et al. determined the radio number of all graphs of order n and diameter $n-2$, where $n \geq 2$ is an integer. Bhatti et al. studied [3] the radio number of wheel-like graphs, while Čada et al. discussed [4] a general version of radio labelings of distance graphs. In [14], Liu gave a lower bound on the radio number for trees and presented a class of trees achieving this bound. In [13], Li et al. determined the radio number for complete m-ary trees. In [10], Halász and Tuza determined the radio number of internally regular complete trees among other things. (A few distance-three labeling problems for such trees of even diameters were studied in [11].) In spite of these efforts, the problem of determining the exact value of the radio number for trees is still open, and it seems unlikely that a universal formula exists for all trees.

Inspired by the work in [13], in this paper we first give a necessary and sufficient condition for a lower bound [14, Theorem 3] (see also Lemma 3.1) on the radio number of trees to be achieved (Theorem 3.2), together with an optimal radio labeling. We also give two sufficient conditions for this bound to be achieved (Theorem 3.6) and obtain an upper bound on the radio number of trees (Theorem 3.7). These results provide methodologies for obtaining the exact values of or upper bounds on the radio number of trees, and using them we determine in Section 4 the radio number for three families of trees, namely banana trees, firecracker trees, and caterpillars in which all vertices on the spine have the same degree. Our result for caterpillars implies the result in [17] for paths. As concluding remarks, in Section 5 we demonstrate that the results on the radio numbers of internally regular complete trees [10, Theorem 1] and complete m-ary trees for $m \geq 3$ [13, Theorem 2] can be obtained by using our method.

2. Preliminaries

We follow [21] for graph-theoretic definition and notation. A *tree* is a connected graph that contains no cycle. In [14] the *weight* of T from $v \in V(T)$ is defined as $w_T(v) = \sum_{u \in V(T)} d(u, v)$ and the *weight* of T as $w(T) = \min\{w_T(v) : v \in V(T)\}$. A vertex $v \in V(T)$ is a *weight centre* [14] of T if $w_T(v) = w(T)$. Denote by $W(T)$ the set of weight centres of T. It was proved in [14, Lemma 2] that every tree T has either one or two weight centres, and T has two weight centres, say, $W(T) = \{w, w'\}$, if and only if w and w' are adjacent and $T - ww'$ consists of two equal-sized components. We view T as rooted at its weight centre $W(T)$: if $W(T) = \{w\}$, then T is rooted at w; if $W(T) = \{w, w'\}$ (where w and w' are adjacent), then T is rooted at w and w' in the sense that both w and w' are at level 0. In either case, if in T the unique path from a weight centre to a vertex $v \notin W(T)$ passes through a vertex u (possibly with $u = v$), then u is called an *ancestor* of v, and v is called a *descendent* of u. If v is a descendent of u and is adjacent to u, then v is a *child* of u. Let $u \notin W(T)$ be adjacent to a weight centre. The subtree induced by u and all its descendent is called a *branch* at u. Two branches are called *different* if they are at two vertices adjacent to the same weight centre, and *opposite* if they are at two vertices adjacent to different weight centres. Note that the latter case occurs only when T has two weight centres. Define

$$L(u) := \min\{d(u, x) : x \in W(T)\}, \quad u \in V(T)$$

to indicate the *level* of u in T. Define the *total level* of T as

$$L(T) := \sum_{u \in V(T)} L(u).$$

For any $u, v \in V(T)$, define

$$\phi(u, v) := \max\{L(x) : x \text{ is a common ancestor of } u \text{ and } v\}$$

$$\delta(u, v) := \begin{cases} 1, & \text{if } W(T) = \{w, w'\} \text{ and } P_{uv} \text{ contains the edge } ww' \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.1. *Let T be a tree with diameter $d \geq 2$. Then for any $u, v \in V(T)$ the following hold:*
(a) $\phi(u, v) \geq 0$;
(b) $\phi(u, v) = 0$ *if and only if u and v are in different or opposite branches;*
(c) $\delta(u, v) = 1$ *if and only if T has two weight centres and u and v are in opposite branches;*
(d) *the distance $d(u, v)$ in T between u and v can be expressed as*

$$d(u, v) = L(u) + L(v) - 2\phi(u, v) + \delta(u, v). \quad (1)$$

3. Radio number of trees

A radio labeling of T is an injective mapping f from $V(T)$ to the set of nonnegative integers; we can always assume that f assigns 0 to some vertex. Thus f induces a linear order of the vertices of T, namely $V(T) = \{u_0, u_1, \ldots, u_{p-1}\}$ (where