ARTICLE IN PRESS

Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On a directed variation of the 1-2-3 and 1-2 Conjectures[☆]

Emma Barme[a], Julien Bensmail[b,*], Jakub Przybyło[c], Mariusz Woźniak[c]

[a] *Laboratoire d'Informatique du Parallélisme, École Normale Supérieure de Lyon, 46 allée d'Italie, 69364 Lyon Cedex 07, France*

[b] *Department of Applied Mathematics and Computer Science, Technical University of Denmark, DK-2800 Lyngby, Denmark*

[c] *AGH University of Science and Technology, al. A. Mickiewicza 30, 30-059 Krakow, Poland*

ARTICLE INFO

Article history:
Received 2 October 2015
Received in revised form 3 August 2016
Accepted 11 August 2016
Available online xxxx

Keywords:
1-2-3 Conjecture
1-2 Conjecture
Digraphs

ABSTRACT

In this paper, we consider the following question, which stands as a directed analogue of the well-known 1-2-3 Conjecture: Given any digraph D with no arc $\overrightarrow{uv}$ verifying $d^+(u) = d^-(v) = 1$, is it possible to weight the arcs of D with weights among $\{1, 2, 3\}$ so that, for every arc $\overrightarrow{uv}$ of D, the sum of incident weights out-going from u is different from the sum of incident weights in-coming to v? We answer positively to this question, and investigate digraphs for which even the weights among $\{1, 2\}$ are sufficient. In relation with the so-called 1-2 Conjecture, we also consider a total version of the problem, which we prove to be false. Our investigations turn to have interesting relations with open questions related to the 1-2-3 Conjecture.

1. Introduction

We here focus on *vertex-distinguishing weightings*, a graph theory notion that attracted more and more attention in the last decade. Basically, given an undirected graph G, the goal is to weight some *elements* of G so that some *well-identified vertices* of G get distinguished relatively to some *aggregate* computed from the weighting. As emphasized in the previous sentence, such problems of correctly weighting a graph are hence made of three main parameters. For any of these variants, the main goal is, given a graph, to deduce the smallest number of consecutive weights $1, \ldots, k$ necessary to obtain a correct distinguishing weighting.

In this paper, we focus on those such problems where edges (among maybe other elements) have to be weighted, and the distinguishing aggregate is the *sum* of weights incident to the vertices. More formally, given an edge-weighting w of some graph G, for every vertex v one may compute[1]

$$\sigma_w(v) := \sum_{u \in N(v)} w(vu),$$

☆ The second author was supported by ERC Advanced Grant GRACOL, project no. 320812, and by ANR grant no. ANR-13-BS02-0007 "STINT". The third author was supported by the National Science Centre, Poland, grant no. 2014/13/B/ST1/01855. The third and fourth authors were partly supported by the Polish Ministry of Science and Higher Education. The fourth author was supported by the National Science Centre, Poland, grant no. DEC-2013/09/B/ST1/01772.

* Corresponding author.
E-mail address: julien.bensmail.phd@gmail.com (J. Bensmail).

1 In case no ambiguity is possible, we will sometimes voluntarily omit the subscript w (*e.g.* write σ for σ_w) to lighten the notations.

http://dx.doi.org/10.1016/j.dam.2016.08.013
0166-218X/

that is, the sum of the weights incident to v. In case w is a total-weighting, every vertex v also has its own weight, which must be involved when computing $\sigma_w(v)$, that is

$$\sigma_w(v) := w(v) + \sum_{u \in N(v)} w(vu)$$

in such a situation. In the setting where σ_w is the distinguishing parameter, three main notions have been studied in the literature:

(1) If *edge-weightings* are considered and *all vertices* of G must be distinguished by σ, the least number of necessary consecutive edge weights is denoted $s(G)$ (and is called the *irregularity strength* of G in the literature).
(2) If *edge-weightings* are considered and only the *adjacent vertices* of G must be distinguished, the least number of necessary consecutive weights is denoted $\chi^e_\Sigma(G)$.
(3) If *total-weightings* are considered and only the *adjacent vertices* of G must be distinguished, the least number of necessary consecutive weights is denoted $\chi^t_\Sigma(G)$.

As we only focus on Items (2) and (3) (that is, on *sum-colouring edge-weighting* and *sum-colouring total-weighting*) in this paper, we will below recall some of their associated backgrounds. For more general details on this wide area (and on the upcoming introductory details), we refer the interested reader to the recent survey by Seamone on this topic [13].

The parameter χ^e_Σ is related to the well-known **1-2-3 Conjecture** raised by Karoński, Łuczak and Thomason [9], which reads as follows (where a *nice graph* refers to a graph with no component isomorphic to K_2).

1-2-3 Conjecture (*Karoński, Łuczak, Thomason [9]*). *For every nice graph G, we have* $\chi^e_\Sigma(G) \leq 3$.

Several constant upper bounds on χ^e_Σ were given towards the 1-2-3 Conjecture, the best one of which being due to Kalkowski, Karoński and Pfender, who proved that $\chi^e_\Sigma(G) \leq 5$ whenever G is nice [8]. Concerning the parameter χ^t_Σ, the following so-called **1-2 Conjecture** was raised by Przybyło and Woźniak [12].

1-2 Conjecture (*Przybyło, Woźniak [12]*). *For every graph G, we have* $\chi^t_\Sigma(G) \leq 2$.

Towards the 1-2 Conjecture, the best known result so far is due to Kalkowski [7], who proved that every graph G verifies $\chi^t_\Sigma(G) \leq 3$.

There have been a few attempts for bringing the 1-2-3 and 1-2 Conjectures to directed graphs, see *e.g.* [1,3,5,10]. Most of all these different directed versions of the 1-2-3 and 1-2 Conjectures were shown to hold, even under strong additional constraints such as list requirements. This results from the fact that these versions, though seemingly close to the 1-2-3 and 1-2 Conjectures in essence, were based on several behaviours that are not so comparable to the ones we have to deal with when considering the original conjectures. Notably, the definitions of some of these versions make the use of induction arguments possible, while such are generally not applicable in the undirected context. This makes us wonder what should be the directed analogues to the 1-2-3 and 1-2 Conjectures that would mimic their behaviours and inherent hardness the best, while fitting to the particularities of the directed context.

In that spirit, we introduce and study new directed analogues of the 1-2-3 and 1-2 Conjectures. Our directed analogue of the 1-2-3 Conjecture is introduced in Section 2, while our analogue of the 1-2 Conjecture is studied in Section 3. We more precisely show our directed analogue of the 1-2-3 Conjecture to be equivalent to solved cases of the 1-2-3 Conjecture, hence giving a positive answer to a question addressed by Łuczak [11]. Using that equivalence, we point out that our directed analogue of the 1-2 Conjecture, though true in specific contexts, is false in general. Unexpected implications of our investigations on the 1-2-3 Conjecture are discussed in Section 4.

2. A Directed 1-2-3 Conjecture

Let D be a simple digraph, and w be an arc-weighting of D. For every vertex v, one can compute two sums incident to v, namely

$$\sigma^-_w(v) := \sum_{u \in N^-(v)} w(\overrightarrow{uv}),$$

i.e. the incident in-coming sum, and

$$\sigma^+_w(v) := \sum_{u \in N^+(v)} w(\overrightarrow{vu}),$$

i.e. the incident out-going sum. We call w *sum-colouring* if, for every arc $\overrightarrow{uv}$ of D, we have

$$\sigma^+_w(u) \neq \sigma^-_w(v).$$

The least number of weights in a sum-colouring k-arc-weighting (if any) of D is denoted $\chi^e_Ł(D)$.

Please cite this article in press as: E. Barme, et al., On a directed variation of the 1-2-3 and 1-2 Conjectures, Discrete Applied Mathematics (2016), http://dx.doi.org/10.1016/j.dam.2016.08.013