ARTICLE IN PRESS

Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Solving vertex coloring problems as maximum weight stable set problems

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ARTICLE INFO

Article history:
Received 19 June 2015
Received in revised form 20 July 2016
Accepted 4 September 2016
Available online xxxx

Keywords:
Vertex coloring
Stable set
Mixed integer linear programming
Computational experiments

ABSTRACT

In Vertex Coloring Problems, one is required to assign a color to each vertex of an undirected graph in such a way that adjacent vertices receive different colors, and the objective is to minimize the cost of the used colors. In this work we solve four different coloring problems formulated as Maximum Weight Stable Set Problems on an associated graph. We exploit the transformation proposed by Cornaz and Jost (2008), where given a graph G, an auxiliary graph $\hat{G}$ is constructed, such that the family of all stable sets of $\hat{G}$ is in one-to-one correspondence with the family of all feasible colorings of G. The transformation in Cornaz and Jost (2008) was originally proposed for the classical Vertex Coloring and the Max-Coloring problems; we extend it to the Equitable Coloring Problem and the Bin Packing Problem with Conflicts. We discuss the relation between the Maximum Weight Stable formulation and a polynomial-size formulation for the VCP, proposed by Campêlo et al. (2008) and called the *Representative formulation*. We report extensive computational experiments on benchmark instances of the four problems, and compare the solution method with the state-of-the-art algorithms. By exploiting the proposed method, we largely outperform the state-of-the-art algorithm for the Max-coloring Problem, and we are able to solve, for the first time to proven optimality, 14 Max-coloring and 2 Equitable Coloring instances.

1. Introduction

In Vertex Coloring Problems, one is required to assign a color to each vertex of an undirected graph in such a way that adjacent vertices receive different colors, and the objective is to minimize the cost of the used colors. In the classical Vertex Coloring Problem (VCP), all colors have the same cost, hence, the objective is to minimize the number of used colors. The Equitable Coloring Problem is a VCP with the additional restriction that subsets of vertices receiving the same color, denoted as *color classes*, differ in cardinality of at most one unit. The Max-coloring Problem is defined as a VCP where each vertex has a positive weight, and the cost of a color is given by the maximum weight of the vertices in the corresponding color class. Finally, by considering a VCP with a positive weight associated with each vertex, and imposing that the total weight of the vertices in a color class does not exceed a given capacity, we obtain a VCP with capacity constraints. Because the problem generalizes the Bin Packing Problem as well, it is known in the literature as Bin Packing Problem with Conflicts. All mentioned problems are NP-hard.

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http://dx.doi.org/10.1016/j.dam.2016.09.018

Coloring problems are very challenging from the computational viewpoint and have several relevant applications, including, just to mention a few, scheduling [17], timetabling [8], frequency assignment [14], register allocation [5] and communication networks [27] (see Malaguti and Toth, [20]).

Problems definition. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A stable set $S \subseteq V(G)$ is a subset of vertices containing no edge, a clique $K \subseteq V(G)$ is a subset of vertices inducing a complete subgraph. The stable sets of G are the cliques of the complementary graph $\overline{G}$ of G.

Given a weight $c \in \mathbb{Z}^{V(G)}$, the *Max Weight Stable Set Problem (MWSSP)* is to determine a stable set S of G maximizing $\sum_{v\in S} c(v)$. We denote by $\alpha(G, c)$ the optimum of MWSSP. The MWSSP can be naturally formulated as a (Mixed-)Integer Linear Program (MIP) since $\alpha(G, c) = \max c^{\top}x$ over x in $STAB(G)$, that is, the set of vectors of $\mathbb{R}^{V(G)}$ satisfying

$$\begin{cases} x_u + x_v \le 1 & uv \in E(G) \\ x_v \in \{0, 1\} & v \in V(G). \end{cases}$$

A *p-coloring of* G is a partition $\mathcal{S} = S_1, \ldots, S_p$ of $V(G)$ into p stable sets S_i, where each stable set represents a color. The *Vertex Coloring Problem (VCP)* is to find a p-coloring of G with a minimum number of colors p. We denote by $\chi(G)$ the optimum of VCP. The VCP is a very challenging problem from the computational viewpoint, for which state-of-the-art exact methods can fail in optimally solving instances with more than 200 vertices. The best performing exact methods for the VCP are based on the Set Covering formulation of the problem, where binary variables are associated with stable sets of G. Since stable sets of an arbitrary graph G are in exponential number with respect to the graph size, Set Covering formulations require column generation techniques and Branch-and-Price algorithms for managing the exponentially many variables. The recent algorithms by Malaguti, Monaci and Toth [19], Gualandi and Malucelli [15], and Held, Cook and Sewell [16], are all very sophisticated implementations of a Branch-and-Price algorithm, embedding specialized (meta)heuric procedures [19], constraint programming techniques [15] and improved algorithms for the column generation subproblem [16]. For random graphs, competitive experimental results are obtained by implicit enumeration schemes based on the DSATUR algorithm by Brélaz [3], see, e.g., [26]. We also mention the Branch-and-Cut approach by Méndez-Díaz and Zabala [23], which is effective for some special classes of graphs.

Given a graph G, a coloring $\mathcal{S} = S_1, \ldots, S_p$ is *equitable* if $|S_i| - |S_j| \le 1$ for each $i, j = 1, \ldots, p$. The *Equitable Coloring Problem* (ECP) is to find an equitable coloring with minimum p. We denote by $\chi_{\text{eq}}(G)$ the optimum of ECP. The most recent mathematical programming contributions to the exact solution of the ECP include the Branch-and-Cut algorithm by Bahiense et al. [1], which exploits an asymmetric formulation; and the MIP formulation by Méndez-Díaz, Nasini and Severín [22], which is strengthened by valid inequalities derived from a polyhedral study, and can be solved directly by a MIP solver. In addition, a DSATUR based Branch-and-Bound algorithm was recently proposed by Méndez-Díaz, Nasini and Severín [21]. To the best of our knowledge, no extended formulation with variables associated with stable sets of G was proposed. A possible reason is the non straightforward definition of equitable cardinality constraints for the color classes in this setting.

Given a graph G and vertex weights $c \in \mathbb{Z}^{V(G)}$, the *Max-coloring Problem* (also known as *Weighted VCP*, see [18,13]) is the problem of determining a coloring $\mathcal{S} = S_1, \ldots, S_p$ of G which minimizes $\psi(\mathcal{S}) := \sum_{i=1}^{p} c_i$ where $c_i = \max_{v\in S_i} c(v)$. We denote by $\chi_{\max}(G, c)$ the optimum of Max-col. When c is a unit vector, $\psi(\mathcal{S}) = p$ and Max-col reduces to the VCP. The best performing exact method for the Max-col is the Branch-and-Price algorithm by Furini and Malaguti [13], which can solve instances with up-to 100 vertices.

Given a graph G, vertex weights $c \in \mathbb{Z}^{V(G)}$, and a nonnegative integer κ, *the Bin Packing Problem with Conflict (BPPC)* is to determine a coloring $\mathcal{S} = S_1, \ldots, S_p$ of G which minimizes p so that the weight of the stable sets $c(S_i) := \sum_{v\in S_i} c(v)$ does not exceed κ. The optimum is denoted by $\chi_{\text{bp}}(G, c, \kappa)$. State-of-the-art algorithms for the BPPC, recently proposed by Fernandes-Muritiba et al. [11], Elhedhli et al. [9], Sadykov and Vanderbeck [25], exploit a Set Covering formulation and implement a Branch-and-Price framework. Apparently, the presence of capacity constraints on the cardinality of the color classes reduces the practical difficulty of solving the BPPC. The mentioned algorithms can solve to optimality instances with up to 500 vertices.

In this work we solve the VCP, ECP, Max-col and BPPC by reformulating them as MWSSPs on an associated graph. We exploit the transformation proposed by Cornaz and Jost [6], where given a graph G, an auxiliary graph $\hat{G}$ is constructed, such that the family of all stable sets of $\hat{G}$ is in one-to-one correspondence with the family of all feasible colorings of G. The transformation was originally proposed for the VCP and for the Max-col. We extend the transformation to the ECP and the BPPC; in these cases additional constraints have to be defined for the MWSSP. The advantage of this approach relies on simplicity: it allows us to solve coloring problems by solving MWSSPs, which can be tackled by problem specific algorithms, or can be formulated as MIPs of polynomial size, and solved by a general purpose MIP solver. We discuss the relation between the MWSSPs reformulation and a polynomial-size formulation for the VCP, proposed by Campêlo, Corrêa and Campos [4] and called the *Representative formulation*. To the best of our knowledge the computational performance of this latter formulation has not been investigated in the literature. Instead, computational results are presented by Bahiense et al. [1], where a Branch-and-Cut algorithm based on the Representative formulation is developed for the Equitable Coloring problem.

The transformation by Cornaz and Jost [6] was exploited by Bonomo, Giandomenico and Rossi to derive polynomial-time algorithms for special classes of graphs [2]. Recently, the same transformation was exploited by Furini, Gabrel and Ternier [12] to derive strong and fast lower bounds for the VCP, which are then used to design a DSATUR based Branch-and-Bound algorithm. The transformation was also exploited in [7] to improve the Theta-Lovász lower bound for VCP.

Please cite this article in press as: D. Cornaz, et al., Solving vertex coloring problems as maximum weight stable set problems, Discrete Applied Mathematics (2016), http://dx.doi.org/10.1016/j.dam.2016.09.018