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# Completely independent spanning trees in some regular graphs

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ABSTRACT

Let $k \geq 2$ be an integer and $T_1, \dots, T_k$ be spanning trees of a graph $G$. If for any pair of vertices $\{u, v\}$ of $V(G)$, the paths between $u$ and $v$ in every $T_i$, $1 \leq i \leq k$, do not contain common edges and common vertices, except the vertices $u$ and $v$, then $T_1, \dots, T_k$ are completely independent spanning trees in $G$. For $2k$-regular graphs which are $2k$-connected, such as the Cartesian product of a complete graph of order $2k-1$ and a cycle, and some Cartesian products of three cycles (for $k = 3$), the maximum number of completely independent spanning trees contained in these graphs is determined and it turns out that this maximum is not always $k$.



## 1. Introduction

Two spanning trees $T_1$ and $T_2$ in $G$ are *edge-disjoint* if $E(T_1) \cap E(T_2) = \emptyset$. For a given tree $T$ and a given pair of vertices $\{u, v\}$ of $T$, let $P_T(u, v)$ be the set of vertices in the unique path between $u$ and $v$ in $T$. Two spanning trees $T_1$ and $T_2$ are *internally vertex-disjoint* if for any pair of vertices $\{u, v\}$ of $V(G)$, $P_{T_1}(u, v) \cap P_{T_2}(u, v) = \{u, v\}$. Finally, the spanning trees $T_1, \dots, T_k$ of $G$ are *completely independent spanning trees* if they are pairwise edge-disjoint and internally vertex-disjoint. Note that for each $i$, $1 \leq i \leq k$, the set of vertices which are not leaves in $T_i$ forms a connected dominating set. Thus, if there exist $k$ internally vertex-disjoint spanning trees in $G$, then there exist $k$ disjoint connected dominating sets.

Disjoint spanning trees have been extensively studied as they are of practical interest for fault-tolerant broadcasting or load-balancing communication systems in interconnection networks: a spanning tree is often used in various network operations; computing completely independent spanning trees guarantees a continuity of service, as each can be immediately used as backup spanning tree if a node or link failure occurs on the current spanning tree. Thus, computing $k$ completely independent spanning trees allows to handle up to $k-1$ simultaneous independent node or link failures. In this context, a network is often modeled by a graph $G$ in which the set of vertices $V(G)$ corresponds to the nodes set and the set of edges $E(G)$ to the set of direct links between nodes.

Completely independent spanning trees were introduced by Hasunuma [9] and then have been studied on different classes of graphs, such as underlying graphs of line graphs [9], maximal planar graphs [10], Cartesian product of two cycles [12] and complete graphs, complete bipartite and tripartite graphs [20]. Moreover, the decision problem that consists in determining if there exist two completely independent spanning trees in a graph $G$ is NP-hard [10]. Recently, sufficient conditions have been determined in order to guarantee the existence of two completely independent spanning trees. These

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conditions are inspired by the sufficient conditions for Hamiltonicity: Dirac's condition [1] and Ore's condition [5]. Moreover, the Dirac's condition has been generalized to more than two trees [4,11,14] and has been independently improved [11,14] for two trees. Also, a recent paper has studied the problem on an interesting class of graphs: the class of $k$-trees, for which the authors have proven that there exist at least $\lceil k/2 \rceil$ completely independent spanning trees [18].

Other works on disjoint spanning trees include independent spanning trees which focus on finding spanning trees $T_1, \ldots, T_k$ rooted at $r$. In independent spanning trees, for any vertex $v$ the paths between $r$ and $v$ in $T_1, \ldots, T_k$ are pairwise internally vertex-disjoint, i.e. for each $i$ and $j$, $1 \leq i < j \leq k$, $P_{T_i}(r, v) \cap P_{T_j}(r, v) = \{r, v\}$. In contrast with the notion of completely independent spanning trees, in independent spanning trees only the paths to $r$ are considered. Thus, $T_1, \ldots, T_k$ may share common vertices or edges, which is not admissible with completely independent spanning trees. Independent spanning trees have been studied in several topologies, including product graphs [19], de Bruijn and Kautz digraphs [6,13], and chordal rings [16]. Related works also include edge-disjoint spanning trees, i.e. spanning trees which are pairwise edge-disjoint only. Edge-disjoint spanning trees have been studied on many classes of graphs, including hypercubes [2], Cartesian product of cycles [3] and Cartesian product of two graphs [15]. Also, there are some works about internally vertex-disjoint spanning trees that are expressed in terms of disjoint connected dominating sets. The maximum number of disjoint connected dominating sets in a graph $G$ is the *connected domatic number* [23]. An interesting result about connected domatic number concerns planar graphs, for which Hartnell and Rall have proven that, except $K_4$ (which has connected domatic number 4), their connected domatic number is bounded by 3 [8]. The problem of constructing a connected dominating set is often motivated by wireless ad-hoc networks [7,22]. Connected dominating sets are used to create a virtual backbone or spine of a wireless ad-hoc network.

We use the following notations: for a tree, a vertex that is not a leaf is called an *inner vertex*. For a vertex $u$ of a graph $G$, let $d_G(u)$ be its degree in $G$, i.e. the number of edges of $G$ incident with it.

For clarity, we recall the definition of the Cartesian product of two graphs: Given two graphs $G$ and $H$, the Cartesian product of $G$ and $H$, denoted $G \square H$, is the graph with vertex set $V(G) \times V(H)$ and edge set $\{(u, u')(v, v') | (u = v \wedge u'v' \in E(H)) \vee (u' = v' \wedge uv \in E(G))\}$.

The following theorem gives an alternative definition [9] of completely independent spanning trees.

**Theorem 1.1** ([9]). *Let $k \geq 2$ be an integer. $T_1, \ldots, T_k$ are completely independent spanning trees in a graph $G$ if and only if they are edge-disjoint spanning trees of $G$ and for any $v \in V(G)$, there is at most one $T_i$ such that $d_{T_i}(v) > 1$.*

It has been conjectured that in any $2k$-connected graph, there are $k$ completely independent spanning trees [10]. This conjecture has been refuted, as there exist $2k$-connected graphs which do not contain two completely independent spanning trees [17,21], for any integer $k$. However, the given counterexamples are not $2k$-regular.

**Proposition 1.2** ([17,21]). *For any $k \geq 2$, there exist $2k$-connected graphs that do not contain two completely independent spanning trees.*

The proof from Péterfalvi [21] or from Kriesell [17] of the previous proposition consists in constructing a $2k$-connected graph with a large proportion of vertices of degree $2k$ adjacent to the same vertices (in the two proofs, the constructed graphs are different) and proving that these vertices of degree $2k$ cannot be all adjacent to inner vertices in a fixed tree. Moreover, Fan et al. [5] have proved that there exist infinitely many 4-connected 4-regular graphs (the 4-connected line graphs of cubic graphs) for which there do not exist two completely independent spanning trees.

This article is organized as follows. Section 2 presents necessary conditions on $2r$-regular graphs in order to have $r$ completely independent spanning trees. Section 3 presents the maximum number of completely independent spanning trees in $K_m \square C_n$, for $n \geq 3$ and $m \geq 3$. In particular, we exhibit the first $2r$-regular graphs, for $r \geq 4$, which are $2r$-connected and which do not contain $r$ completely independent spanning trees. In Section 4, we determine three completely independent spanning trees in some Cartesian products of three cycles $C_{n_1} \square C_{n_2} \square C_{n_3}$, for $3 \leq n_1 \leq n_2 \leq n_3$.

## 2. Necessary conditions on $2r$-regular graphs

We begin with some notation.

**Definition 2.1.** Let $T$ be a spanning tree of a graph $G$ of order $n$. Let $G'$ be a $2r$-regular graph for which there exist $r$ completely independent spanning trees $T_1, \ldots, T_r$. We define the following notation:

- $\mathrm{IN}(T)$ is the set of inner vertices of $T$;
- the *potential extra degree* of the spanning tree $T$ is $\mathrm{ped}(T) = |\mathrm{IN}(T)|r - n + 2$;
- $E^l(G') = E(G') \setminus \bigcup_{1 \leq i \leq r} E(T_i)$;
- $E^l_{T_i}(G')$ is the subset of the edges of $E^l(G')$ that have their two extremities in $\mathrm{IN}(T_i)$, i.e. $E^l_{T_i}(G') = \{uv \in E(G) | u, v \in \mathrm{IN}(T_i),\ uv \notin E(T_i)\}$, for $i \in \{1, \ldots, r\}$;

Notice that, by definition, for any spanning tree $T$, we have $\mathrm{ped}(T) \geq 0$ and that, the number of inner vertices of $T$ of degree at most $r$ is bounded by $\mathrm{ped}(T)$.

In the following proposition, we illustrate basic facts about the introduced notation.

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