ARTICLE IN PRESS
Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮
Contents lists available at ScienceDirect
Discrete Applied Mathematics
journal homepage: www.elsevier.com/locate/dam

Harmonious colourings of graphs

Ewa Drgas-Burchardt *, Katarzyna Gibek

Faculty of Mathematics, Computer Science and Econometrics, University of Zielona Góra, ul. prof. Z.Szafrana 4a, 65-516 Zielona Góra, Poland

ARTICLE INFO

Article history:
Received 18 August 2015
Received in revised form 28 July 2016
Accepted 4 September 2016
Available online xxxx

Keywords:
Nordhaus–Gaddum type theorem
Graph polynomials
Harmonious colourings

ABSTRACT

A λ-*harmonious colouring* of a graph G is a mapping from $V(G)$ into $\{1, \ldots, \lambda\}$ that assigns colours to the vertices of G such that each vertex has exactly one colour, adjacent vertices have different colours, and any two edges have different colour pairs. The *harmonious chromatic number* $h(G)$ of a graph G is the least positive integer λ, such that there exists a λ-harmonious colouring of G.

Let $h(G, \lambda)$ denote the number of all λ-harmonious colourings of G. In this paper we analyse the expression $h(G, \lambda)$ as a function of a variable λ. We observe that this is a polynomial in λ of degree $|V(G)|$, with a zero constant term. Moreover, we present a reduction formula for calculating $h(G, \lambda)$. Using reducing steps we show the meaning of some coefficients of $h(G, \lambda)$ and prove the Nordhaus–Gaddum type theorem, which states that for a graph G with diameter greater than two

$$h(G) + \frac{1}{2}\chi(\overline{G^2}) \leq |V(G)|,$$

where $\chi(\overline{G^2})$ is the chromatic number of the complement of the square of a graph G. Also the notion of harmonious uniqueness is discussed.

1. Introduction

The first paper on some variant of harmonious colouring appeared in 1982 [11] on the occasion of studying some problems connected to register allocation for computer programs and assignment of frequencies to radio stations. The definition of this notion, in the form used in the paper, was announced in 1983 [13].

A *harmonious colouring* of a graph G is a mapping that assigns colours to the vertices of G such that each vertex has exactly one colour, adjacent vertices have different colours, and any two edges have different colour pairs. The *harmonious chromatic number* $h(G)$ of a graph G is the minimum number of colours assigned to the vertices in a harmonious colouring of G.

Let $\chi(G)$ denote the chromatic number of G and let G^2 denote the graph resulting from G by the addition of edges joining vertices which are at distance two in G (the square of G). Observe that if two different vertices of a graph have common neighbour, then they obtain different colours in any harmonious colouring. Using just introduced notions, this fact can be formulated in the following way.

Remark 1. For each n-vertex graph G it holds $h(G) \geq \chi(G^2)$. In particular, if G has diameter at most two, then $h(G) = n$.

* Corresponding author.
E-mail addresses: E.Drgas-Burchardt@wmie.uz.zgora.pl (E. Drgas-Burchardt), katarzynagibek@wp.pl (K. Gibek).

http://dx.doi.org/10.1016/j.dam.2016.09.017
0166-218X/

It was shown by Hopcroft and Krishnamoorthy [13] that the problem of determining the harmonious chromatic number of a graph is NP-hard. Moreover, Edwards and McDiarmid [10] showed that the problem remains hard even restricted to the class of trees. On the other hand, the harmonious chromatic number is known for almost all graphs. Indeed, almost all graphs have diameters less than three and for each such a graph its harmonious chromatic number and the number of vertices are equal (see Remark 1).

The wide literature connected with problems of graph harmonious colourability contains mainly papers in which the lower/upper bounds on the harmonious chromatic number or the exact value of this number for some special classes of graphs are given [1–4,7–9,16,18]. Also the subject of complexity of related problems is very popular [5,6]. Let $\overline{G}$ stand for the graph with the same vertex-set as G, in which two vertices are adjacent if and only if they are non-adjacent in G (the complement of G). The only known Nordhaus–Gaddum type theorem in this field says that the sum $h(G)+h(\overline{G})$ does not exceed twice the number of vertices of a graph G [15]. We recall that the famous Nordhaus–Gaddum theorem [17] applied to the square of a graph G states $\chi(G^2)+\chi(\overline{G^2}) \le |V(G)|+1$. Noting that $h(G) \ge \chi(G^2)$ (see Remark 1) we formulate the main result of this paper as Theorem 7.

Theorem 7. *For each n-vertex graph G with diameter greater than two it holds*

$$h(G)+\frac{1}{2}\chi(\overline{G^2}) \le n.$$

The proof of this result uses the reducing step of the reduction formula for calculating the number of all harmonious colourings of a graph with λ accepted colours (Theorem 4). We prove that the function that counts such colourings is a polynomial in λ and we describe many of its properties (Theorems 1–3).

2. Preliminaries

Throughout this paper, we consider finite and undirected graphs G with vertex set $V(G)$ and edge set $E(G)$ that are loopless and have no multiple edges. In general, we follow the notation and terminology of [19].

In particular, the *distance* between two vertices u, v in a graph G is the length of a shortest $(u-v)$-path in G (the number of edges in the path). The *diameter* of a graph G is the largest distance between two vertices taken over all pairs of vertices of G. For a set $S \subseteq V(G)$ by $G[S]$ we denote a subgraph of G induced by S. The equality $G=H$ means that graphs G and H are isomorphic. By $G-e$ we denote a graph resulting from G by the removal of one edge. As we mentioned before, the symbols $\overline{G}$ and G^2 stand for the complement and the square of a graph G, respectively. A complete graph, a path, a cycle of order n are denoted K_n, P_n, C_n. A complete bipartite graph with the cardinalities of partite sets p, q is denoted by $K_{p,q}$. For any graph G and any positive integer k the notation kG is used for the disjoint union of k copies of G.

The symbols $\mathbb{N}$ and $\mathbb{N}_0$ stand for the set of positive and nonnegative integers. For $a, b \in \mathbb{N}_0$ we adopt the convention $[a,b]=\{a,\dots,b]$ with the assumption $[a,b]=\emptyset$ when $b<a$ and with the simplification $[1,b]=[b]$. Additionally, for $\lambda \in \mathbb{R}$, $n \in \mathbb{N}$, we use the notation $[\lambda]_n=\lambda(\lambda-1)\cdots(\lambda-n+1)$ and denote by $\binom{X}{n}$ the set of all n-element subsets of the set X. If $\lambda \in \mathbb{N}$, then by a λ-*colouring of* X we mean a mapping $g: X \to [\lambda]$. For an arbitrary graph G, a λ-colouring g of $V(G)$ is called *proper* if g assigns different colours to adjacent vertices. The chromatic number of G, denoted $\chi(G)$, is the least positive integer λ such that there exists a proper λ-colouring of G. If a proper λ-colouring g of G additionally satisfies the condition $\{g(v_1),g(v_2)\} \ne \{g(x_1),g(x_2)\}$ for any two edges $v_1v_2, x_1x_2 \in E(G)$, then it is called a λ-*harmonious* colouring of G or shortly a *harmonious* colouring of G. By $h(G,\lambda)$ we denote the number of all λ-harmonious colourings of a graph G. The harmonious chromatic number of G, denoted by $h(G)$, is the least positive integer λ, such that there exists a λ-harmonious colouring of G.

In many cases, we consider graphs G whose edge set is partitioned as $E(G)=E_1(G) \cup E_2(G)$ in a way to be specified later. We refer to such a graph G as a *graph with two kinds of edges*. The complement $\overline{G}$ in this case will then have the usual meaning, namely, $V(\overline{G})=V(G)$ and $E(\overline{G})=\{xy: \ x,y \in V(G), \ xy \notin E_1(G) \cup E_2(G)\}$.

3. Harmonious partitions

Let $j \in \mathbb{N}$ and let G be a graph. A partition of $V(G)$ into nonempty parts $V_1,\dots,V_j$ such that $G[V_i]$ is an edgeless graph for each $i \in [j]$ and for any two different $i_1, i_2 \in [j]$ the graph $G[V_{i_1} \cup V_{i_2}]$ has at most one edge is called *harmonious*.

Lemma 1. *Let $n \in \mathbb{N}$, G be an n-vertex graph and $V' \subseteq V(G)$.*

1. *If the partition of $V(G)$ in which V' is one of the partition parts is harmonious, then $G^2[V']$ is an edgeless graph.*
2. *If $G^2[V']$ is an edgeless graph, then the partition of $V(G)$ in which V' is one of the parts and remaining parts are one-element sets is harmonious.*

Proof. From the definition of a harmonious partition, the vertices in V' are pairwise nonadjacent in G and no two of them have common neighbour in G. It means that $G^2[V']$ is edgeless.

Please cite this article in press as: E. Drgas-Burchardt, K. Gibek, Harmonious colourings of graphs, Discrete Applied Mathematics (2016), http://dx.doi.org/10.1016/j.dam.2016.09.017