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Chinese Postman Problem on edge-colored multigraphs

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ABSTRACT

It is well-known that the Chinese Postman Problem on undirected and directed graphs is polynomial-time solvable. We extend this result to edge-colored multigraphs. Our result is in sharp contrast to the Chinese Postman Problem on mixed graphs, i.e., graphs with directed and undirected edges, for which the problem is NP-hard.

1. Introduction

In this paper, we consider edge-colored multigraphs. In such multigraphs, each edge is assigned a color; a multigraph G is called *k-edge-colored* if only colors from $[k] := \{1, 2, \ldots, k\}$ are used in G. A walk[1] W in an edge-colored multigraph is called *properly colored (PC)* if no two consecutive edges of W have the same color. PC walks are of interest in graph theory applications, e.g., in genetic and molecular biology [17,20,19], in design of printed circuit and wiring boards [21], and in channel assignment in wireless networks [2,18]. They are also of interest in graph theory itself as generalizations of walks in undirected and directed graphs. Indeed, if we assign different colors to all edges of an undirected multigraph, every walk not traversing the same edge twice becomes PC. Also, consider the standard transformation from a directed graph D into a 2-edge-colored graph G by replacing every arc uv of D by a path with a blue edge uw_{uv} and a red edge $w_{uv}v$, where w_{uv} is a new vertex [3]. Clearly, every directed walk in D corresponds to a PC walk in G (with end-vertices in $V(G)$) and vice versa. There is an extensive literature on PC walks; for a detailed survey of pre-2009 publications, see Chapter 16 of [3], more recent papers include [1,7,12–14].

A walk is *closed* if it starts and ends in the same vertex. (A closed walk W has no last edge, and every edge in W has a following edge; if W is PC, each edge of W is of different color to the following edge.) An *Euler trail* in a multigraph G is a closed walk which traverses each edge of G exactly once. PC Euler trails were one of the first types of PC walks studied in the literature and the first papers that studied PC Euler trails were motivated by theoretical questions [6,11] as well as questions in molecular biology [17]. To formulate a characterization of edge-colored graphs with PC Euler trails by Kotzig [11], we introduce additional terminology. A vertex in an edge-colored multigraph is *balanced* if no color appears on more than half

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[1] Terminology on walks used in this paper is given in the next section.

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of the edges incident with the vertex, and *even* if it is of even degree. We say that an edge-colored graph is *PC Euler* if it contains a PC Euler trail.

Theorem 1 ([11]). *An edge-colored multigraph G is PC Euler if and only if G is connected and every vertex of G is balanced and even.*

Benkouar et al. [4] described a polynomial-time algorithm to find a PC Euler trail in an edge-colored multigraph, if it contains one. Studying DNA physical mapping, Pevzner [16] came up with a simpler polynomial-time algorithm solving the same problem.

In this paper, we consider the *Chinese Postman Problem on edge-colored graphs (CPP-ECG)*: *given a connected edge-colored multigraph G with non-negative weights on its edges, find a PC closed walk in G which traverses all edges of G and has the minimum weight*[2] *among such walks*.

Observe that to solve CPP-ECG, it is enough to find a PC Euler edge-colored multigraph G^* of minimum weight such that $V(G^*) = V(G)$ and for every pair of distinct vertices u, v and color i, G^* has $p^* > 0$ parallel edges between vertices u and v of color i if and only if G has at least one and at most p^* edges of color i between u and v. (To find the actual walk, we can use the algorithm from [4] or [17].)

CPP-ECG is a generalization of the PC Euler trail problem as an instance G has a PC Euler trail if and only if $G^* = G$. CPP-ECG is also a generalization of the Chinese Postman Problem (CPP) on both undirected and directed multigraphs (the arguments are the same as for PC walks above). However, while CPP on both undirected and directed multigraphs has a solution on every connected multigraph G, it is not the case for CPP-ECG. Indeed, there is no solution on any connected edge-colored multigraph containing a vertex incident to edges of only one color.

It is not hard to solve CPP on undirected and directed multigraphs [10]. For a directed multigraph G, we construct a flow network N by assigning lower bound 1, upper bound ∞ and cost $\omega(uv)$ to each arc uv, where $\omega(uv)$ is the weight of uv in G. A minimum-cost circulation in N viewed as an Euler directed multigraph corresponds to a CPP solution and vice versa. For an undirected multigraph G, we construct an edge-weighted complete graph H whose vertices are odd degree vertices of G and the weight of an edge xy in H equals the minimum weight path between x and y in G. Now find a minimum-weight perfect matching M in H and add to G a minimum-weight path of G between x and y for each edge xy of M. The resulting Euler multigraph corresponds to a CPP solution and vice versa.

We will prove that CPP-ECG is polynomial-time solvable as well. Note that our proof is significantly more complicated than that for CPP on undirected and directed graphs. As in the undirected case, we construct an auxiliary edge-weighted complete graph H and seek a minimum-weight perfect matching M in it. However, the construction of H and the arguments justifying the appropriate use of M are significantly more complicated. This can partially be explained by the fact that CPP-ECG has no solution on many edge-colored multigraphs.

Note that there is another generalization of CPP on both undirected and directed multigraphs, namely, CPP on mixed multigraphs, i.e., multigraphs that may have both edges and arcs. However, CPP on mixed multigraphs is NP-hard [15]. It is fixed-parameter tractable when parameterized by both number of edges and arcs [24,8] and W[1]-hard when parameterized by pathwidth [9]. For more information on the classical and parameterized complexity of CPP and its generalizations, see the excellent survey by van Bevern et al. [24].

2. Preliminaries

Walks. A *walk* in a multigraph is a sequence $W = v_1e_1v_2 \dots v_{p-1}e_{p-1}v_p$ of alternating vertices and edges such that vertices v_i and v_{i+1} are end-vertices of edge e_i for every $i \in [p-1]$. A walk W is *closed* (*open*, respectively) if $v_1 = v_p$ ($v_1 \neq v_p$, respectively). A *trail* is a walk in which all edges are distinct.

For technical reasons we will consider walks with fixed end vertices and call them *fixed end-vertex (FEV) walks*. Note that an open walk is necessarily an FEV walk since the end-vertices are predetermined, whereas any vertex in a closed walk can be viewed as its two end-vertices and thus fixing such a vertex is somewhat similar to assigning a root vertex in a tree. An FEV walk $W = v_1e_1v_2 \dots v_{p-1}e_{p-1}v_p$ is *PC* in an edge-colored graph if the colors of e_i and e_{i+1} are different for every $i \in [p-2]$. Note that we do not require that colors of e_{p-1} and e_1 are different even if $v_1 = v_p$. Thus, a PC FEV walk might not be a PC walk if $v_1 = v_p$.

Let $e = xy$ be an edge in an edge-colored multigraph G. The operation of *double subdivision* of e replaces e with an (x, y)-path P_e with three edges such that the weight of P_e equals that of edge e.

It is easy to see that in our study of PC walks, we may restrict ourselves to graphs rather than multigraphs. Indeed, it suffices to double subdivide every parallel edge e and assign the original color of e to the first and third edges of P_e and a new color to the middle edge.

Finding PC FEV walks. Let $\mathbb{R}_+$ denote the set of non-negative real numbers. To show a polynomial-time algorithm for CPP-ECG, we will use the following:

[2] The weight of a walk is the sum of the weights of its edges.