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Bounds on the independence number of a graph in terms of order, size and maximum degree

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ABSTRACT

The independence number of a graph G is the maximum cardinality of an independent set of vertices in G. In this paper we obtain several new lower bounds for the independence number of a graph in terms of its order, size and maximum degree, and characterize graphs achieving equalities for these bounds. Our bounds improve previous bounds for graphs with large maximum degree.

1. Introduction

In this paper we study independence number and transversal number in graphs. For notation and terminology not presented here we refer to [9]. Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. We denote by $n(G)$ and $m(G)$, or just n, m if G is specified, the order and size of G, respectively. For a vertex $v \in V(G)$, let $N_G(v) = \{u \mid uv \in E(G)\}$ denote the *open neighborhood* of v. The *degree* of a vertex v, $\deg_G(v)$, or just $\deg(v)$, in a graph G denotes the number of neighbors of v in G. We denote by $\Delta(G)$ and $\delta(G)$ the maximum degree and the minimum degree of the vertices of G, respectively. If S is a subset of $V(G)$, then we let $\delta_G[S] = \min\{\deg_G(v) \mid v \in S\}$. For a subset S of $V(G)$, we denote by $G[S]$ the subgraph of G induced by S. A *clique* is a subset of vertices such that its induced subgraph is complete. The *clique number*, $\omega(G)$, of a graph G is the number of vertices in a maximum clique in G. A *leaf* in a graph is a vertex of degree one, and a *support vertex* is one that is adjacent to a leaf. An edge of G is called a *pendant edge* if at least one of its vertices is a leaf of G. An *isolated vertex* in a graph is a vertex that is not adjacent to any vertex. We use the standard notation $[k] = \{1, 2, \dots, k\}$.

A set S of vertices in a graph G is an *independent set* if no pair of vertices of S is adjacent. The *independence number* of G, denoted by $\alpha(G)$, is the maximum cardinality of an independent set in G. An independent set of cardinality $\alpha(G)$ is called an $\alpha(G)$-set. A vertex *covers* an edge if it is incident with the edge. A *transversal* in the graph G is a set of vertices that covers all the edges. We remark that a transversal is also called a *vertex-cover* in the literature. The *transversal number* of G, denoted by $\tau(G)$, is the minimum cardinality of a transversal in G. A transversal of cardinality $\tau(G)$ is called a $\tau(G)$-set. Since an isolated vertex covers no edge, we consider graphs without isolated vertices. The following are well-known.

Observation 1. *For any graph G of order n, $\alpha(G) + \tau(G) = n$.*

Observation 2. *For every graph G of order n and size m, $\tau(G) \leqslant (n + m)/3$.*

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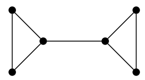

Fig. 1. Double-paw.

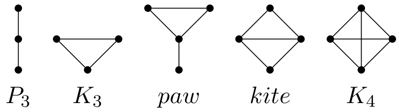

Fig. 2. The family $\mathcal{G}_1$.

From Observations 1 and 2 we obtain that for any graph G of order n and size m,

$$\alpha(G) \geqslant \frac{1}{3}(2n - m). \tag{1}$$

The independence number is one of the most fundamental and well-studied graph parameters (see, for example, [1,2,4–8,12]). Several improvements of (1) have been presented by several authors, (see, for example [3,10,11,13–15,17]).

Recently, Löwenstein, Pedersen, Rautenbach, and Regen [16] established the following lower bound on the independence number of a graph.

Theorem 3 (*Löwenstein, Pedersen, Rautenbach and Regen, [16]*)**.** *If G is a connected graph of order n and size m, then $\alpha(G) \geqslant 2n/3 - m/4 - 1/3$.*

Henning and Löwenstein [10] improved the bound of Theorem 3 for graphs not belonging to a specific family of graphs $\mathcal{G}$ (see Section 2 of [10]).

Theorem 4 (*Henning and Löwenstein, [10]*)**.** *If $G \notin \mathcal{G}$ is a connected graph of order n and size m, then $\alpha(G) > 2n/3 - m/4$.*

Our aim in this paper is to present new lower bounds for the independence number of a graph by considering the vertices of maximum degree. We also characterize graphs achieving equalities for the presented bounds. Our results improve Theorems 3 and 4 for graphs with large maximum degree.

In this paper, for a subset S of vertices of G, we denote by $G - S$ the graph obtained from G by removal of S and also removal of all isolated vertices in $G[V(G)\backslash S]$. If $S = \{v\}$, we denote $G - v$ instead of $G - S$ for convenience. Let C_n, P_n and K_n be the cycle, the path, and the complete graph on n vertices, respectively. The following observation will be used repeatedly in the following sections.

Observation 5. *For the complete graph K_n and the path P_n and the cycle C_n, $\tau(K_n) = n - 1$, $\tau(P_n) = \lceil (n-1)/2 \rceil$ and $\tau(C_n) = \lceil n/2 \rceil$.*

2. Families of graphs

In this section we introduce some families of graphs which we shall use in the following sections. We refer to the graph shown in Fig. 1 as a *double-paw*. Let $\mathcal{G}_1$ be the family of graphs shown in Fig. 2.

Given a family $\mathcal{F}$ of graphs, for an integer $1 \leqslant t \leqslant \min\{\omega(F) \mid F \in \mathcal{F}\}$, we define a new family $\mathcal{F}[t]$ of graphs as follows. A graph G belongs to $\mathcal{F}[t]$ if and only if G can be obtained from a sequence $F_1, F_2, \ldots, F_k \in \mathcal{F}$, (not necessarily distinct), for some integer k, by coinciding a clique of order t in each of F_i, for $i \in [k]$. The graph F_i, $i \in [k]$, will be referred as the $F_i(G)^t$*-unit*, or just the $F_i(G)$*-unit* if t is clear. We denote by $Q_t(G)$ the coincided clique of order t in G. Note that $Q_t(G)$ is isomorphic to a clique of order t in every F_i, for $i \in [k]$, and for convenience, we assume that $Q_t(G)$ is a clique in every F_i, for $i \in [k]$. If $\mathcal{F} = \{F\}$, then we denote $F[t]$, rather than $\mathcal{F}[t]$. Fig. 3 illustrates three graphs in the family $\mathcal{G}_1[1]$. In the graph G_1, the P_3-paths with vertex sets $\{a, b, c\}$ and $\{a, d, e\}$ are the $P_3(G_1)$-units, the kite with vertex set $\{a, f, g, h\}$ is the $kite(G_1)$-unit, the paw with vertex set $\{a, i, j, k\}$ is the $paw(G_1)$-unit, and the complete graph with vertex set $\{a, l, n, m\}$ is the $K_4(G_1)$-unit. Note that $Q_1(G_1) = \{a\}$.

Let $\mathcal{G}_{1,1}$ be the family of graphs $G \in \mathcal{G}_1[1]$ such that G has at least one $F(G)$-unit for $F \in \{K_3, K_4\}$, and the vertex in $Q_1(G)$ has minimum degree in each $F(G)$-unit for $F \in \mathcal{G}_1$. Note that the vertex in $Q_1(G)$ has maximum degree in G, since G has at least one $F(G)$-unit for $F \in \{K_3, K_4\}$. Observe that the graph G_1 shown in Fig. 3 belongs to $\mathcal{G}_{1,1}$ as well, while the graphs G_2 and G_3 do not belong to $\mathcal{G}_{1,1}$.

For each positive integer t, we define a family $\mathcal{G}_{i,t}$, $i \in \{2, 3, 4, 5, 6\}$ as follows.

- Let $\mathcal{G}_{2,t}$ be the family of graphs $G \in \{F, K_{t+1}\}[t]$, where $F \in \{K_{t+5}\} \cup K_{t+4}[t+1]$, such that G has precisely one $K_{t+1}(G)$-unit and one $F(G)$-unit, and $Q_t(G) \subseteq Q_{t+1}(F)$ if $F \neq K_{t+5}$. Two graphs in the family $\mathcal{G}_{2,1}$, are illustrated in Fig. 4.

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