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Circular-arc hypergraphs: Rigidity via connectedness

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ABSTRACT

A *circular-arc hypergraph* $\mathcal{H}$ is a hypergraph admitting an *arc ordering*, that is, a circular ordering of the vertex set $V(\mathcal{H})$ such that every hyperedge is an arc of consecutive vertices. We give a criterion for the uniqueness of an arc ordering in terms of connectedness properties of $\mathcal{H}$. This generalizes the relationship between rigidity and connectedness disclosed by Chen and Yesha (1991) in the case of interval hypergraphs. Moreover, we state sufficient conditions for the uniqueness of *tight* arc orderings where, for any two hyperedges A and B such that $A \subseteq B \neq V(\mathcal{H})$, the corresponding arcs must share a common endpoint. We notice that these conditions are obeyed for the closed neighborhood hypergraphs of proper circular-arc graphs, implying for them the known rigidity results that were originally obtained using the theory of local tournament graph orientations.

1. Introduction

1.1. Interval and circular-arc hypergraphs

An *interval ordering* of a hypergraph $\mathcal{H}$ with a finite vertex set $V = V(\mathcal{H})$ is a linear ordering $v_1, \ldots, v_n$ of V such that every hyperedge of $\mathcal{H}$ is an interval of consecutive vertices. This notion can be generalized to an *arc ordering* $v_1, \ldots, v_n$ where the vertices are *circularly ordered* (i.e., v_1 succeeds v_n) so that every hyperedge is an *arc* of consecutive vertices.

An *interval hypergraph* is a hypergraph admitting an interval ordering. Similarly, if a hypergraph admits an arc ordering, we call it *circular-arc* (using also the shorthand *CA*). In the terminology stemming from computational genomics, interval hypergraphs are exactly those hypergraphs whose incidence matrix has the *consecutive ones property*; see, e.g., [7]. Similarly, a hypergraph is CA exactly when its incidence matrix has the *circular ones property*; see [8,18] for the relevance to computational genomics and [10,11] for the algorithmic aspects.

Our goal is to study the conditions under which interval and circular-arc hypergraphs are *rigid* in the sense that they have a unique interval or arc ordering, respectively. Since any interval (or arc) ordering can be changed to another interval (or arc) ordering by reversing, we always mean uniqueness *up to reversal*. An obvious necessary condition for being rigid is that a hypergraph has no *twins*, that is, no two vertices such that every hyperedge contains either both or none of them.

We say that two sets A and B *overlap* and write $A \between B$, if A and B have nonempty intersection and neither of the two sets includes the other. To facilitate notation, we use the same character $\mathcal{H}$ to denote a hypergraph and the set of its hyperedges. We call $\mathcal{H}$ *overlap-connected* if the graph $(\mathcal{H}, \between)$ is connected. A vertex of $\mathcal{H}$ is *isolated* if it is not contained in any hyperedge. As a starting point, we refer to the following rigidity result.

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Theorem 1.1 (*Chen and Yesha [4]*). *A twin-free overlap-connected interval hypergraph without isolated vertices has a unique interval ordering (up to reversal).*

If we want to extend this result to CA hypergraphs, the property of being overlap-connected does obviously not suffice. For example, the twin-free hypergraph $\mathcal{H} = \{\{a, b\}, \{a, b, c\}, \{b, c, d\}\}$ is overlap-connected but has essentially different arc orderings. Hence, we need to assume a stronger kind of connectedness. When A and B are overlapping subsets of V (i.e., $A \between B$) that additionally satisfy $A \cup B \neq V$, we say that A and B *strictly overlap* and write $A \between^* B$.

Quilliot [19] proves that a CA hypergraph $\mathcal{H}$ on n vertices has a unique arc ordering if and only if for every set $X \subset V(\mathcal{H})$ with $1 < |X| < n - 1$ there exists a hyperedge $H \in \mathcal{H}$ such that $H \between^* X$. Note that this criterion is not efficiently verifiable (at least not directly) as it involves quantification over exponentially many subsets X.

We call a hypergraph $\mathcal{H}$ *strictly overlap-connected* if the graph $(\mathcal{H}, \between^*)$ is connected. The treatment of interval hypergraphs in [4] can easily be adapted for proving a sufficient rigidity condition for CA hypergraphs: *A twin-free, strictly overlap-connected CA hypergraph has a unique arc representation (up to reversal)*. Moreover, in Section 2 we prove the following criterion.

Theorem 1.2. *Given a CA hypergraph $\mathcal{H}$ on $n \geq 4$ vertices, let $\mathcal{H}'$ be the hypergraph on the same vertex set obtained from $\mathcal{H}$ by removing all hyperedges of size 1, $n - 1$, and n. Then $\mathcal{H}$ has a unique arc ordering (up to reversal) if and only if $\mathcal{H}'$ is twin-free and strictly overlap-connected.*

1.2. Tight orderings

Let us denote by $A \bowtie B$ that two sets A and B have a non-empty intersection. By the standard terminology, a hypergraph $\mathcal{H}$ is *connected* if the graph $(\mathcal{H}, \bowtie)$ is connected. Note that the assumption made in Theorem 1.1 cannot be weakened just to connectedness; consider $\mathcal{H} = \{\{a\}, \{a, b, c\}\}$ as the simplest example. Thus, if we want to weaken the assumption, we have also to weaken the conclusion.

Call an arc ordering of a hypergraph $\mathcal{H}$ *tight* if, for any two hyperedges A and B such that $A \subseteq B \neq V$, the corresponding arcs share an endpoint.[2] The definition of a *tight interval ordering* is similar: We require that the intervals corresponding to hyperedges A and B share an endpoint whenever $A \subseteq B$ (the condition $B \neq V$ is now dropped as the complete interval V has two endpoints, while the complete arc V has none). The class of hypergraphs admitting a tight interval ordering is characterized in terms of forbidden subhypergraphs in [17] (for interval hypergraphs, such a characterization is known due to [23]). Tight orderings inherently appear in the study of proper interval and proper circular-arc graphs; see the next subsection.

Let A and B be nonempty sets. Note that $A \bowtie B$ iff $A \between B$ or $A \subseteq B$ or $A \supseteq B$. Likewise, we define

$$A \bowtie^* B, \quad \text{if } A \between^* B \text{ or } A \subseteq B \text{ or } A \supseteq B,$$

and say that A and B *strictly intersect*. We call a hypergraph $\mathcal{H}$ *strictly connected* if the graph $(\mathcal{H}, \bowtie^*)$ is connected. In Section 2 we show that the approach of Chen and Yesha [4] works as well for tight orderings.

Theorem 1.3. 1. *A twin-free connected hypergraph without isolated vertices has at most one tight interval ordering (up to reversal).*
2. *A twin-free, strictly connected hypergraph has at most one tight arc ordering (up to reversal).*

1.3. Neighborhood hypergraphs of PCA graphs

For a vertex v of a graph G, the set of vertices adjacent to v is denoted by $N(v)$. Furthermore, $N[v] = N(v) \cup \{v\}$. We define the *neighborhood hypergraph* of G by $\mathcal{N}(G) = \{N(v)\}_{v \in V(G)}$ and the *closed neighborhood hypergraph* of G by $\mathcal{N}[G] = \{N[v]\}_{v \in V(G)}$.

An *interval* (resp. *arc*) *representation* of a graph G is a mapping from the vertex set of G to an interval (resp. CA) hypergraph $\mathcal{H}$ such that two vertices of G are adjacent exactly when the corresponding hyperedges of $\mathcal{H}$ have a nonempty intersection. Such a representation is *proper* if none of two hyperedges of $\mathcal{H}$ includes the other. Graphs having such representations are called *proper interval* and *proper circular-arc (PCA) graphs*. Roberts [20] discovered that G is a proper interval graph if and only if $\mathcal{N}[G]$ is an interval hypergraph. The case of PCA graphs is more complex. If G is a PCA graph, then $\mathcal{N}[G]$ is a CA hypergraph. The converse is not always true. The graphs whose closed neighborhood hypergraphs are circular-arc are known as *concave-round graphs* [1], and they contain PCA graphs as a proper subclass. Taking a closer look at the relationship between PCA graphs and CA hypergraphs, Tucker [24] distinguishes the case when the complement graph $\overline{G}$ is non-bipartite and shows that then G is PCA exactly when $\mathcal{N}[G]$ is CA. In general, G is a PCA graph if and only if the hypergraph $\mathcal{N}[G]$ has a tight arc ordering; cf. [15].

[2] The *endpoints* of an arc $A \neq V$ are the two uniquely determined elements a^- and a^+ of V such that A is the chain with respect to the successor relation on V starting in a^- and ending in a^+.