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Exploring the complexity of the integer image problem in the max-algebra

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ABSTRACT

We investigate the complexity of the problem of finding an integer vector in the max-algebraic column span of a matrix, which we call the integer image problem. We show some cases where we can determine in strongly polynomial time whether such an integer vector exists, and find such an integer vector if it does exist. On the other hand we also describe a group of related problems each of which we prove to be NP-hard. Our main results demonstrate that the integer image problem is equivalent to finding a special type of integer image of a matrix satisfying a property we call *column typical*. For a subclass of matrices this problem is polynomially solvable but if we remove the column typical assumption then it becomes NP-hard.

1. Introduction

This paper deals with the task of finding integer vectors in the max-algebraic column span of a matrix. For $a, b \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$ we define $a \oplus b = \max(a, b)$, $a \otimes b = a + b$ and extend the pair $(\oplus, \otimes)$ to matrices and vectors in the same way as in linear algebra, that is (assuming compatibility of sizes)

$$(\alpha \otimes A)_{ij} = \alpha \otimes a_{ij},$$
$$(A \oplus B)_{ij} = a_{ij} \oplus b_{ij}, \quad \text{and}$$
$$(A \otimes B)_{ij} = \bigoplus_k a_{ik} \otimes b_{kj}.$$

Except for complexity calculations, all multiplications in this paper are in max-algebra and we will usually omit the $\otimes$ symbol. Note that α^{-1} stands for $-\alpha$, and we will use ε to denote $-\infty$ as well as any vector or matrix whose every entry is $-\infty$. The zero vector is denoted by $\mathbf{0}$. A vector/matrix whose every entry belongs to $\mathbb{R}$ is called *finite*. If a matrix has no ε rows (columns) then it is called *row* (*column*) $\mathbb{R}$-*astic* and it is called *doubly* $\mathbb{R}$-*astic* if it is both row and column $\mathbb{R}$-astic.

Max-algebra (also called tropical algebra) is a rapidly expanding area of idempotent mathematics, linear algebra and applied discrete mathematics. One key advantage is that problems from areas such as operational research, science and engineering which are non-linear in the conventional algebra, can be modeled as linear problems within the max-algebraic setting [1,7,9,10]. Applications of max-algebra are both theoretical and practical; in [10] the Dutch railway system is modeled using max-algebra.

The *integer image problem* (IIM) is the problem of determining whether there is an integer vector in the column span (called here the image space) of a matrix $A \in \overline{R}^{m\times n}$. The set of integer images is

$$IIm(A) := \{z \in \mathbb{Z}^m : (\exists x \in \overline{R}^n) Ax = z\}.$$

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We also define $X(A)$ to be the set of vectors x for which Ax belongs to the set of integer images, that is

$$X(A) := \{x \in \overline{R}^n : Ax \in IIm(A)\}.$$

A related question is whether $X(A) \cap \overline{Z}^n$ is nonempty, we define

$$IIm^*(A) := \{z \in \mathbb{Z}^m : (\exists x \in \overline{Z}^n)Ax = z\}.$$

One application of the integer image problem is as follows [6]. Suppose machines $M_1, \ldots, M_n$ produce components for products $P_1, \ldots, P_m$. Let x_j denote the starting time of M_j and a_{ij} be the time taken for M_j to complete its component for P_i. Then all components for product P_i are ready at completion time

$$c_i = \max(a_{i1} + x_1, \ldots, a_{in} + x_n) \quad i = 1, \ldots, m.$$

Equivalently this can be written as $Ax = c$. In this context the integer image problem asks whether there exists a set of starting times for which the completion times are integer (this can easily be extended to ask for any discrete set of values). If we additionally require that the starting times are integer/discrete values then we want to find $c \in IIm^*(A)$.

Further it is known [4] that the *max-algebraic integer eigenspace*, defined as

$$\{x \in \mathbb{Z}^n : Ax = \lambda x, x \neq \varepsilon\}$$

for a fixed eigenvalue $\lambda \in \mathbb{R}$, is equal to the integer image space of a matrix B obtained from A. Currently it is not known whether it is possible to find an integer eigenvector in polynomial time. The eigenproblem in max-algebra can be used to analyze stability in production systems [2,7]: Assume machines $M'_1, \ldots, M'_n$ work interactively and in stages. In each stage all the machines produce components for the other machines to use in the next stage. Let $x_i(r)$ denote the starting time of the rth stage on machine M'_i and a_{ij} denote the time taken for M'_j to complete its component for machine M'_i. Then

$$x_i(r+1) = \max(x_1(r) + a_{i1}, \ldots, x_n(r) + a_{in}) \quad i = 1, \ldots, n, \ r = 0, 1, \ldots$$

This can be written as $x(r+1) = Ax(r)$. A *steady regime* is reached if this process moves forwards in regular steps, i.e. if $x(r+1) = \lambda x(r)$ for all $r \geq 0$. Clearly this occurs if and only if $x(0)$ is an eigenvector of A corresponding to some eigenvalue $\lambda \in \mathbb{R}$. It is natural to look for integer starting times, and therefore we aim to solve the integer eigenproblem. A solution to the corresponding integer image problem would achieve this.

An algorithm for testing whether $IIm(A) \neq \emptyset$ and finding an integer image if it exists was described in [4]. This algorithm always terminates in a finite number of steps and is pseudopolynomial if the input matrix is finite. We investigate whether the problem could in fact be in P, the class of polynomially solvable problems.

In searching for integer solutions to the integer image problem one helpful tool is being able to identify potential active positions. Given vectors x, z such that $Ax = z$ we say that a position (i, j) is *active* with respect to x or z if $a_{ij} + x_j = z_i$, and *inactive* otherwise. It will be useful in this paper to talk about the entries of the matrix corresponding to active positions and therefore we say that an element a_{ij} of A is *active* if and only if the position (i, j) is active. In the same way we call a column A_j *active* if it contains an active position.

We define a *column typical* matrix to be a matrix $A \in \overline{\mathbb{R}}^{m\times n}$ such that for each j we have $fr(a_{ij}) \neq fr(a_{kj})$ for any i and $k, i \neq k$ such that $a_{ij}, a_{kj} \in \mathbb{R}$. Note that $fr(\cdot)$ denotes the fractional part and will be defined in Section 2. Observe that, given a column typical matrix, there can be at most one active entry in each column with respect to any integer image. This significantly reduces the set of candidates for active entries, and it is therefore expected that the integer image problem for column typical matrices will be easier to solve than for general matrices.

In this paper we will consider a number of integer image problems, each with an additional requirement on the set of integer images. These are detailed in the definition below. Fig. 1 outlines the relations between these problems.

Definition 1.1. Given $A \in \overline{R}^{m\times n}$ we consider the following problems related to the Integer Image Problem.
(IIM–CT) If A is column typical does there exist $x \in \overline{R}^n$ such that $Ax \in \mathbb{Z}^m$?
(IIM–CT-P1) If A is column typical does there exist $x \in \overline{R}^n$ such that $Ax \in \mathbb{Z}^m$ with exactly one active entry per row with respect to x?
(IIM-P1) Does there exist $x \in \overline{R}^n$ such that $Ax \in \mathbb{Z}^m$ with exactly one active entry per row with respect to x?
(IIM*) Does there exist $x \in \overline{Z}^n$ such that $Ax \in \mathbb{Z}^m$?

In Section 2 we summarise the existing theory necessary for the presentation of our results and describe some simple cases for which we can determine whether $IIm(A) \neq \emptyset$ in strongly polynomial time, and find an integer image if it exists. These cases include IIM–CT for square matrices. In Section 3 we give two different transformations of a general matrix $A \in \overline{R}^{m\times n}$ to a matrix $B \in \overline{R}^{m\times mn}$ for which $IIm(A) = IIm(B)$ and give reasons why we suspect $IIm(B)$ will be easier to describe than $IIm(A)$. In particular determining whether $IIm(B) \neq \emptyset$ reduces to checking whether B is a yes instance of IIM–CT or IIM* and for both these problems we can find an integer image in a special case. However, in general B fails to satisfy the requirements of this special case so this does not solve the integer image problem, but it does lend support to the idea that the integer image problem could be solvable in (strongly) polynomial time. In Section 4 we show that IIM is polynomially equivalent to IIM–CT-P1 and IIM–CT. Section 5 contains the proof that IIM-P1 is NP-hard. Since the only