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The *a*-graph coloring problem

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ABSTRACT

No proof of the 4-color conjecture reveals why it is true; the goal has not been to go beyond proving the conjecture. The standard approach involves constructing an unavoidable finite set of reducible configurations to demonstrate that a minimal counterexample cannot exist. We study the 4-color problem from a different perspective. Instead of planar triangulations, we consider near-triangulations of the plane with a face of size 4; we call any such graph an *a*-graph. We state an *a*-graph coloring problem equivalent to the 4-color problem and then derive a coloring condition that a minimal *a*-graph counterexample must satisfy, expressing it in terms of equivalence classes under Kempe exchanges. Through a systematic search, we discover a family of *a*-graphs that satisfy the coloring condition, the fundamental member of which has order 12 and includes the Birkhoff diamond as a subgraph. Higher-order members include a string of Birkhoff diamonds. However, no member has an applicable parent triangulation that is internally 6-connected, a requirement for a minimal counterexample. Our research suggests strongly that the coloring and connectivity conditions for a minimal counterexample are incompatible; infinitely many *a*-graphs meet one condition or the other, but we find none that meets both.

1. Graph terminology

We use standard graph terminology. All graphs considered in this article are assumed to be planar unless otherwise stated. $V(G)$ denotes the *vertex set* of the graph G. Two vertices are *adjacent* if they share an edge. Such an edge is said to be *incident* to the two vertices it joins. A *triangulation* is a graph in which all faces are delineated by three edges and a *near-triangulation* is a graph in which all faces but one are delineated by three edges. In this article, we refer to the edges delineating the sole non-triangular face in a near-triangulation as the *boundary* of the graph and the vertices on the boundary as *boundary vertices*. Vertices not on the boundary are referred to as *interior vertices*. An *internal path* is one having no edge on the boundary. A *separating n-cycle* in G has vertices of G both inside and outside the n-cycle. A connected graph is *k-connected* if it has more than k vertices and remains connected whenever fewer than k vertices are deleted. In an *r-regular* graph, all vertices have degree r. A *proper vertex-coloring* of G is a coloring of $V(G)$ in which no two adjacent vertices have the same color. The only graph colorings we consider are proper vertex-colorings; we often refer to them merely as *colorings*. We use the numbers 1, 2, 3, 4,…, to denote the different colors available for coloring a graph. We often use the term *4-coloring* to mean any coloring using no more than 4 colors. The expression $c(w)$ translates as *the color of vertex w in the coloring c*. A *Kempe chain* is a maximal connected subgraph of G whose vertices in a coloring of G use only two colors. A Kempe chain

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that uses the colors j and k is referred to as a *j-k chain*. A vertex w colored j that is not adjacent to any vertex colored $k \neq j$ is a *single-vertex* or *short j-k Kempe chain*. Exchanging colors j and k for such a chain is the same as changing the color of w to k.

For purposes of this article, a *Kempe exchange* is a recoloring of G in which the colors j and k are exchanged in a designated, non-empty, proper subset of all j-k Kempe chains. For example, if G has three pairwise disjoint 1–2 Kempe chains, then exchanging colors on any one or any two counts as a single Kempe exchange.

2. Introduction

This article is primarily about discovering why the 4-color conjecture is true. No existing proof [1,2,8,9] accomplishes that — it was never the goal. Succeeding at the endeavor may open up a new avenue by which the 4-color conjecture can ultimately be proved without relying on a computer. Even if that turns out not to be the case, we contend that it is worthwhile trying to understand what it is that renders the 4-color problem solvable.

In its tightest formulation, the statement of the 4-color problem is to show that any planar triangulation has a 4-coloring. Instead of planar triangulations, we study near-triangulations of the plane in which the sole non-triangular face has size 4. We call any such near-triangulation an *a-graph* because it is an "almost-triangulated-graph". Instead of the 4-color problem, we study the *a*-graph coloring problem.

The *a*-graph coloring problem. Let xy be any edge in an arbitrary planar triangulation T. Show that the a-graph $G = T - xy$ has a 4-coloring c in which $c(x) \neq c(y)$.

The 4-color problem and the a-graph coloring problem are trivially equivalent. Start with an uncolored T and delete the edge xy, give the resulting G a coloring c that solves the a-graph coloring problem, then replace the edge xy to obtain a 4-coloring of T. Conversely, start with an uncolored G, insert the edge xy, give the resulting T a coloring c that solves the 4-color problem, then delete the edge xy to obtain the required 4-coloring of G.

What can possibly be achieved by this near sleight-of-hand? What is the motivation for studying the a-graph coloring problem? The answers lie in examining all distinct colorings of both T and G. Obviously, all colorings of T are colorings of G. But there are colorings c of G in which $c(x) = c(y)$ that are not colorings of T. Generally, many of these will be Kempe-equivalent to colorings of T. This observation underpins our approach. *The key concept is to obtain a proper coloring of some target graph, say the triangulation T, by using colorings of a related graph, say the a-graph G, that are not proper colorings of T but are Kempe-equivalent to proper colorings of T*. Suppose that any triangulation of lower order than T has a 4-coloring, either because T is a minimal counterexample or purely as an inductive assumption. By contracting the edge xy in T rather than deleting it, we obtain a triangulation that can be given a 4-coloring c. Upon reversing the contraction without restoring the xy edge, we arrive at G with 4-coloring c in which $c(x) = c(y)$ and can then proceed to analyze the a-graph coloring problem, attempting to navigate by means of Kempe exchanges from c to a 4-coloring c' in which $c'(x) \neq c'(y)$.

It is useful to enhance our notation regarding the designation and description of an a-graph G. We establish the convention of always drawing G with the 4-face as an exterior face and orienting that 4-face as shown in Fig. 6.1, labeling the boundary cycle $uxvy$, thus establishing (x, y) and (u, v) as the two pairs of *opposite (non-adjacent) boundary vertices*. Then we can refer to the two *parent triangulations* of any a-graph G as $G + xy$ and $G + uv$. We always denote the left-hand vertex on the 4-face by x, the right-hand vertex by y, the bottom vertex by u, and the top vertex by v.

With respect to graph structure, the same G results whether the edge xy is deleted in the parent $G + xy$ or the edge uv in the parent $G + uv$. For some purposes, it will suffice to discuss G without reference to a specific parent triangulation, but for many purposes it will not, and in those situations the *applicable parent* needs to be designated. When that is the case, G is best thought of as two different graphs, one with parent $G + xy$ and one with parent $G + uv$. In the first case, we refer to the *parented a-graph* as G_{xy}, and in the second case, we refer to the *parented a-graph* as G_{uv}. A helpful reminder as to which parent is applicable is to think of G_{xy} as having a *ghost edge xy* and of G_{uv} as having a *ghost edge uv*.

Clearly the parented a-graphs G_{xy} and G_{uv} can differ as to the connectivity of their respective applicable parents and we shall see that they can also differ as to how their equivalence classes under Kempe exchanges are categorized, both matters crucial to determining whether they can be minimal a-graph counterexamples. So, for example, G_{xy} might be a minimal a-graph counterexample while G_{uv} is not. Thus the need to distinguish between them. When the a-graph coloring problem for G applies to the (x, y) pair, we refer to G as G_{xy}, and when the a-graph coloring problem for G applies to the (u, v) pair, we refer to G as G_{uv}. When we talk about the structure of an a-graph without reference to a particular pair of opposite boundary vertices, we simply refer to the a-graph as G. For instance, we might simply say that G has minimum degree 4.

3. Overview

In line with other work, ours is based on the notion of a minimal counterexample. To be a minimal counterexample to the 4-color conjecture, a planar triangulation must be *internally 6-connected*. An internally 6-connected triangulation has minimum degree 5 and no separating 3-cycles or 4-cycles. Further, if any separating 5-cycle is removed, at least one of the components created has only a single vertex. A lucid description of this property can be found in both [8,9]. There are an infinite number of internally 6-connected planar triangulations, each of which is potentially a minimal counterexample.