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Note

Interval edge-colorings of composition of graphs

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ABSTRACT

An edge-coloring of a graph G with consecutive integers $c_1, \dots, c_t$ is called an *interval t-coloring* if all colors are used, and the colors of edges incident to any vertex of G are distinct and form an interval of integers. A graph G is interval colorable if it has an interval t-coloring for some positive integer t. The set of all interval colorable graphs is denoted by $\mathfrak{N}$. In 2004, Giaro and Kubale showed that if $G, H \in \mathfrak{N}$, then the Cartesian product of these graphs belongs to $\mathfrak{N}$. In the same year they formulated a similar problem for the composition of graphs as an open problem. Later, in 2009, the second author showed that if $G, H \in \mathfrak{N}$ and H is a regular graph, then $G[H] \in \mathfrak{N}$. In this paper, we prove that if $G \in \mathfrak{N}$ and H has an interval coloring of a special type, then $G[H] \in \mathfrak{N}$. Moreover, we show that all regular graphs, complete bipartite graphs and trees have such a special interval coloring. In particular, this implies that if $G \in \mathfrak{N}$ and T is a tree, then $G[T] \in \mathfrak{N}$.

1. Introduction

All graphs considered in this paper are finite, undirected, and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. For a graph G, by $\overline{G}$ we denote the complement of the graph G. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$, the maximum degree of G by $\Delta(G)$, and the chromatic index of G by $\chi'(G)$. The terms and concepts that we do not define can be found in [1,8,20,34].

A proper edge-coloring of a graph G is a coloring of the edges of G such that no two adjacent edges receive the same color. A proper edge-coloring of a graph G with consecutive integers $c_1, \dots, c_t$ is an *interval t-coloring* if all colors are used, and the colors of edges incident to each vertex of G form an interval of integers. A graph G is *interval colorable* if it has an interval t-coloring for some positive integer t. The set of all interval colorable graphs is denoted by $\mathfrak{N}$. The concept of interval edge-coloring of graphs was introduced by Asratian and Kamalian [2] in 1987. In [2], they proved that if $G \in \mathfrak{N}$, then $\chi'(G) = \Delta(G)$. Asratian and Kamalian also proved [2,3] that if a triangle-free graph G admits an interval t-coloring, then $t \le |V(G)| - 1$. In [16,17], Kamalian investigated interval colorings of complete bipartite graphs and trees. In particular, he proved that the complete bipartite graph $K_{m,n}$ has an interval t-coloring if and only if $m + n - \gcd(m, n) \le t \le m + n - 1$, where $\gcd(m, n)$ is the greatest common divisor of m and n. In [24], Petrosyan investigated interval colorings of complete graphs and hypercubes. In particular, he proved that if $n \le t \le \frac{n(n+1)}{2}$, then the hypercube Q_n has an interval t-coloring. Later, in [27], it was shown that the hypercube Q_n has an interval t-coloring if and only if $n \le t \le \frac{n(n+1)}{2}$. In [31], Sevast'janov

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proved that it is an NP-complete problem to decide whether a bipartite graph has an interval coloring or not. In papers [2,3,6,7,9,16,17,20,24,26–28,31], the problems of the existence, construction and estimating of the numerical parameters of interval colorings of graphs were investigated. Surveys on this topic can be found in some books [1,15,20].

Graph products [8] were first introduced by Berge [5], Sabidussi [30], Harary [10] and Vizing [32]. In particular, Sabidussi [30] and Vizing [32] showed that every connected graph has a unique decomposition into prime factors with respect to the Cartesian product. In the same direction there are also many interesting problems of decomposing of the different products of graphs into Hamiltonian cycles. In particular, in [4] it was proved Bermond's conjecture that states: if two graphs are decomposable into Hamiltonian cycles, then their composition is decomposable, too. A lot of work was done on various topics related to graph products, on the other hand there are still many questions open. For example, it is still open Hedetniemi's conjecture [12], Vizing's conjecture [33] and the conjecture of Harary, Kainen and Schwenk [11].

There are many papers [13,14,19,21–23,29,35] devoted to proper edge-colorings of various products of graphs, however very little is known on interval colorings of graph products. Interval colorings of Cartesian products of graphs were first investigated by Giaro and Kubale [6]. In [7], Giaro and Kubale proved that if $G, H \in \mathfrak{N}$, then $G\square H \in \mathfrak{N}$. In 2004, they formulated [20] a similar problem for the composition of graphs as an open problem. In 2009, the second author [25] showed that if $G, H \in \mathfrak{N}$ and H is a regular graph, then $G[H] \in \mathfrak{N}$. Later, Yepremyan [28] proved that if G is a tree and H is either a path or a star, then $G[H] \in \mathfrak{N}$. Some other results on interval colorings of various products of graphs were obtained in [20, 25–28].

In this paper, we prove that if $G \in \mathfrak{N}$ and H has an interval coloring of a special type, then $G[H] \in \mathfrak{N}$. Moreover, we show that all regular graphs, complete bipartite graphs and trees have such a special interval coloring. In particular, this implies that if $G \in \mathfrak{N}$ and T is a tree, then $G[T] \in \mathfrak{N}$.

2. Notations, definitions and auxiliary results

We use standard notations C_n and K_n for the simple cycle and complete graph on n vertices, respectively. We also use standard notations $K_{m,n}$ and $K_{m,n,l}$ for the complete bipartite and tripartite graphs, respectively, one part of which has m vertices, the other part has n vertices and the third part has l vertices.

For two positive integers a and b with $a \leq b$, we denote by $[a, b]$ the interval of integers $\{a, \ldots, b\}$.

Let $L = (l_1, \ldots, l_k)$ be an ordered sequence of nonnegative integers. The smallest and largest elements of L are denoted by $\underline{L}$ and $\overline{L}$, respectively. The length (the number of elements) of L is denoted by $|L|$. By $L(i)$, we denote the ith element of L $(1 \leq i \leq k)$. An ordered sequence $L = (l_1, \ldots, l_k)$ is called a *continuous sequence* if it contains all integers between $\underline{L}$ and $\overline{L}$. If $L = (l_1, \ldots, l_k)$ is an ordered sequence and p is nonnegative integer, then the sequence $(l_1 + p, \ldots, l_k + p)$ is denoted by $L \oplus p$. Clearly, $(L \oplus p)(i) = L(i) + p$ for any $p \in \mathbb{Z}_+$.

Let G and H be two graphs. The composition (lexicographic product) $G[H]$ of graphs G and H is defined as follows:

$$V(G[H]) = V(G) \times V(H),$$
$$E(G[H]) = \{(u_1, v_1)(u_2, v_2) \colon u_1u_2 \in E(G) \vee (u_1 = u_2 \wedge v_1v_2 \in E(H))\}.$$

A *partial edge-coloring* of G is a coloring of some of the edges of G such that no two adjacent edges receive the same color. If α is a proper edge-coloring of G and $v \in V(G)$, then $S(v, \alpha)$ (*spectrum* of a vertex v) denotes the set of all colors appearing on edges incident to v. The smallest and largest colors of $S(v, \alpha)$ are denoted by $\underline{S}(v, \alpha)$ and $\overline{S}(v, \alpha)$, respectively. A proper edge-coloring α of G with consecutive integers $c_1, \ldots, c_t$ is called an *interval t-coloring* if all colors are used, and for any $v \in V(G)$, the set $S(v, \alpha)$ is an interval of integers. A graph G is *interval colorable* if it has an interval t-coloring for some positive integer t. The set of all interval colorable graphs is denoted by $\mathfrak{N}$. For a graph $G \in \mathfrak{N}$, the smallest and the largest values of t for which it has an interval t-coloring are denoted by $w(G)$ and $W(G)$, respectively.

In [2,3], Asratian and Kamalian obtained the following result.

Theorem 1. *If $G \in \mathfrak{N}$, then $\chi'(G) = \Delta(G)$. Moreover, if G is a regular graph, then $G \in \mathfrak{N}$ if and only if $\chi'(G) = \Delta(G)$.*

In [16], Kamalian proved the following result on complete bipartite graphs.

Theorem 2. *For any $m, n \in \mathbb{N}$, the complete bipartite graph $K_{m,n}$ is interval colorable, and*

(1) $w\left(K_{m,n}\right) = m + n - \gcd(m, n)$,
(2) $W\left(K_{m,n}\right) = m + n - 1$,
(3) *if $w\left(K_{m,n}\right) \leq t \leq W\left(K_{m,n}\right)$, then $K_{m,n}$ has an interval t-coloring.*

In [18], König proved the following result on bipartite graphs.

Theorem 3. *If G is a bipartite graph, then $\chi'(G) = \Delta(G)$.*