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Note

Upper bounds for the achromatic and coloring numbers of a graph ☆

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ABSTRACT

Dvořák et al. introduced a variant of the Randić index of a graph G, denoted by $R'(G)$, where $R'(G) = \sum_{uv\in E(G)} \frac{1}{\max\{d(u),d(v)\}}$, and $d(u)$ denotes the degree of a vertex u in G. The coloring number $col(G)$ of a graph G is the smallest number k for which there exists a linear ordering of the vertices of G such that each vertex is preceded by fewer than k of its neighbors. It is well-known that $\chi(G) \le col(G)$ for any graph G, where $\chi(G)$ denotes the chromatic number of G. In this note, we show that for any graph G without isolated vertices, $col(G) \le 2R'(G)$, with equality if and only if G is obtained from identifying the center of a star with a vertex of a complete graph. This extends some known results. In addition, we present some new spectral bounds for the coloring and achromatic numbers of a graph.

1. Introduction

We consider finite simple graphs. Let $G = (V(G), E(G))$ be a graph. For a vertex $v \in V(G)$, $N_G(v)$ denotes the set of vertices adjacent to v in G. The *degree* of v in G, denoted by $d_G(v)$ (or simply by $d(v)$), is the number of edges of G incident with v. Since G is simple, $d_G(v) = |N_G(v)|$. A vertex of degree zero is called an *isolated vertex*. As usual, $\delta(G)$ and $\Delta(G)$ denote the minimum degree and the maximum degree of G, respectively. The Randić index $R(G)$ of a (molecular) graph G was introduced by Milan Randić [19] in 1975 as the sum of $1/\sqrt{d(u)d(v)}$ over all edges uv of G, where $d(u)$ denotes the degree of a vertex u in G. Formally,

$$R(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d(u)d(v)}}.$$

This index is useful in mathematical chemistry and has been extensively studied, see [15]. For some recent results on the Randić index, we refer to [3,17,18,16].

The harmonic index of a graph G, denoted by $H(G)$, is another vertex-degree-based topological index, and was defined in [8] as follows:

$$H(G) = \sum_{uv\in E(G)} \frac{2}{d(u)+d(v)}.$$

Some more recent work on the Harmonic index and related vertex-degree-based topological indices can be found in [6,10,26].

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In 2011 Dvořák et al. [4] introduced a variation of the Randić index of a graph G, denoted by $R'(G)$, which has been further studied by Knor et al. [14]. Formally,

$$R'(G) = \sum_{uv \in E(G)} \frac{1}{\max\{d(u), d(v)\}}.$$

It is clear from the definitions that for a graph G,

$$R'(G) \le H(G) \le R(G). \tag{1}$$

The chromatic number of G, denoted by $\chi(G)$, is the smallest number of colors needed to color all vertices of G such that no pair of adjacent vertices is colored the same. The coloring number $col(G)$ of a graph G is the least integer k such that G has a vertex ordering in which each vertex is preceded by fewer than k of its neighbors. The *degeneracy* of G, denoted by $deg(G)$, is defined as $deg(G) = \max\{\delta(H) : H \subseteq G\}$. It is well-known (see Page 8 in [13]) that for any graph G,

$$col(G) = deg(G) + 1. \tag{2}$$

List coloring is an extension of coloring of graphs, introduced by Vizing [23] and independently, by Erdős et al. [7]. For each vertex v of a graph G, let $L(v)$ denote a list of colors assigned to v. A list coloring is a coloring l of vertices of G such that $l(v) \in L(v)$ and $l(x) \ne l(y)$ for any $xy \in E(G)$, where $v, x, y \in V(G)$. A graph G is k-choosable if for any list assignment L to each vertex $v \in V(G)$ with $|L(v)| \ge k$, there always exists a list coloring l of G. The list chromatic number $\chi_l(G)$ (or choice number) of G is the minimum k for which G is k-choosable.

It is well known that for any graph G,

$$\chi(G) \le \chi_l(G) \le col(G) \le \Delta(G) + 1. \tag{3}$$

The detail of the inequalities in (3) can be found in a survey paper by Tuza [22] on list coloring.

In 2009, Hansen and Vukičević [11] established the following relation between the Randić index and the chromatic number of a graph.

Theorem 1.1 (*Hansen and Vukičević [11]*)**.** *Let G be a simple graph with chromatic number $\chi(G)$ and Randić index $R(G)$. Then $\chi(G) \le 2R(G)$ and equality holds if G is a complete graph, possibly with some additional isolated vertices.*

Some interesting extensions of Theorem 1.1 were recently obtained.

Theorem 1.2 (*Deng et al. [2]*)**.** *For a graph G, $\chi(G) \le 2H(G)$ with equality if and only if G is a complete graph possibly with some additional isolated vertices.*

Theorem 1.3 (*Wu, Yan and Yang [25]*)**.** *If G is a graph of order n without isolated vertices, then*

$$col(G) \le 2R(G),$$

with equality if and only if $G \cong K_n$.

Let n and k be two integers such that $n \ge k \ge 1$. We denote the graph obtained from identifying the center of the star $K_{1,n-k}$ with a vertex of the complete graph K_k by $K_k \bullet K_{1,n-k}$. In particular, if $k \in \{1, 2\}$, $K_k \bullet K_{1,n-k} \cong K_{1,n-1}$; if $k = n$, $K_k \bullet K_{1,n-k} \cong K_n$. The primary aim of this note is to prove stronger versions of Theorems 1.1–1.3, noting the inequalities in (1).

Theorem 1.4. *For a graph G of order n without isolated vertices, $col(G) \le 2R'(G)$, with equality if and only if $G \cong K_k \bullet K_{1,n-k}$ for some $k \in \{1, \dots, n\}$.*

Corollary 1.5. *For a graph G of order n without isolated vertices, $\chi(G) \le 2R'(G)$, with equality if and only if $G \cong K_k \bullet K_{1,n-k}$ for some $k \in \{1, \dots, n\}$.*

Corollary 1.6. *For a graph G of order n without isolated vertices, $\chi_l(G) \le 2R'(G)$, with equality if and only if $G \cong K_k \bullet K_{1,n-k}$ for some $k \in \{1, \dots, n\}$.*

Corollary 1.7. *For a graph G of order n without isolated vertices, $col(G) \le 2H(G)$, with equality if and only if $G \cong K_n$.*

The proofs of these results will be given in the next section.

Recall that a *k-coloring* of a graph G is a mapping $c : V(G) \mapsto \{1, 2, \dots, k\}$ such that no two adjacent vertices are assigned the same color. A *complete k-coloring* of a graph G is a k-coloring of the graph such that for each pair of different colors there are adjacent vertices with these colors. The *achromatic number* of G, denoted by $\psi(G)$, is the maximum number k for which the graph has a complete k-coloring. Clearly, $\chi(G) \le \psi(G)$ for a graph G. In general, $col(G)$ and $\psi(G)$ are incomparable. Tang et al. [21] proved that for a graph G, $\psi(G) \le 2R(G)$.

In Section 3, we prove new bounds for the coloring and achromatic numbers of a graph in terms of its spectrum, which strengthen $col(G) \le 2R(G)$ and $\psi(G) \le 2R(G)$. In Section 4, we provide an example and propose two related conjectures.