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Paired many-to-many disjoint path covers of hypertori

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ABSTRACT

Let n be a positive integer, and let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ be an n-tuple of integers such that $d_i \geq 2$ for all i. A hypertorus $Q_n^{\mathbf{d}}$ is a simple graph defined on the vertex set $\{(v_1, v_2, \ldots, v_n) : 0 \leq v_i \leq d_i - 1 \text{ for all } i\}$, and has edges between $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ if and only if there exists a unique i such that $|u_i - v_i| = 1$ or $d_i - 1$, and for all $j \neq i$, $u_j = v_j$; a two-dimensional hypertorus $Q_2^{\mathbf{d}}$ is simply a torus. In this paper, we prove that if $d_1 \geq 3$ and $d_2 \geq 3$, then $Q_2^{\mathbf{d}}$ is balanced paired 2-to-2 disjoint path coverable if both d_i are even, and is paired 2-to-2 disjoint path coverable otherwise. We also discuss a connection between this result and the popular game Flow Free. Finally, we prove several related results in higher dimensions.

1. Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. A *path of length* n is an alternating sequence $v_1, e_1, v_2, e_2, \ldots, e_{n-1}, v_n$ of vertices and edges such that all vertices v_i, $i = 1, \ldots, n$, are distinct; if instead $v_1 = v_n$, then we call this sequence a *closed path* or *cycle*. A *hamiltonian path* (respectively *hamiltonian cycle*) is a path (cycle) that includes all vertices of G. A graph G is said to be *hamiltonian* if it contains a hamiltonian cycle, and is said to be *hamiltonian connected* if there exists a hamiltonian path between any two distinct vertices in G ([14]; see also e.g. [11]). A bipartite graph G with partite sets V_1 and V_2 is said to be *hamiltonian laceable* if:

- $|V_1| = |V_2|$, and there exists a hamiltonian path between any pair of vertices u in V_1 and v in V_2, or
- $|V_1| = |V_2| + 1$, and there exists a hamiltonian path between any pair of (distinct) vertices u and v in V_1.

Let n be a positive integer, and $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ an n-tuple of integers such that $d_i \geq 2$ for all $i = 1, 2, \ldots, n$. Let $L_n^{\mathbf{d}}$ denote an *n-dimensional rectangular lattice*, which is defined to be the graph on the vertex set

$$\{(v_1, v_2, \ldots, v_n) : 0 \leq v_i \leq d_i - 1 \text{ for all } i = 1, 2, \ldots, n\}$$

with edges between $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ if and only if there exists a unique i, $1 \leq i \leq n$, such that $|u_i - v_i| = 1$, and for all $j \neq i$, $u_j = v_j$. In 1977, Simmons proved the following theorem.

Theorem 1.1 ([18]). *Every n-dimensional lattice, $n \geq 3$, is hamiltonian laceable.*

Two-dimensional rectangular lattices appear, for example, in the popular game Flow Free [2]. This game is played on a d_1 by d_2 rectangular board, which is equivalent to the rectangular lattice $L_2^{(d_1,d_2)}$. In this game, the player must connect each given pair of dots using a path such that every square is covered and no path crosses itself or any other path. For example, Fig. 1 shows a game board along with one possible solution.

However, it should not be surprising that many boards in two dimensions will not have a solution. For example, in Fig. 2(a), no solution exists and any attempt results in a situation similar to the one pictured in Fig. 2(b).

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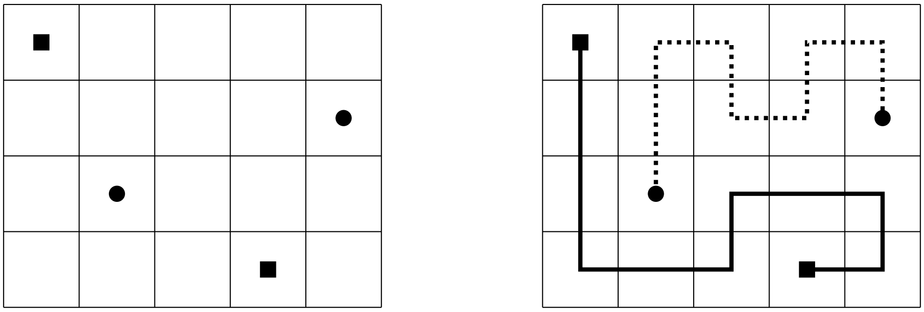

Fig. 1. A Flow Free game on a rectangular lattice along with one possible solution.

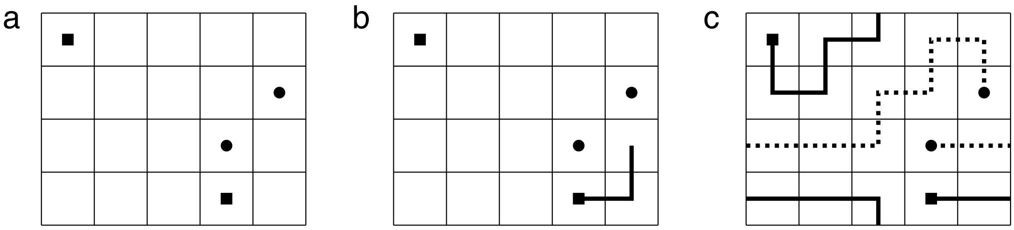

Fig. 2. A game (Fig. 2(a)) which has no solution when played on a rectangular lattice (Fig. 2(b)). However, this game does have a solution when played on a torus (Fig. 2(c)).

This problem occurs because while rectangular lattices are natural objects, they are irregular in the sense that not all vertices are of the same degree. Hence, we will now consider the same game on a torus, meaning that opposite edges are identified and a path can exit one edge of the board and reenter at the corresponding square on the opposite edge. Notice that the game in Fig. 2(a) can now be solved in this environment (see Fig. 2(c)). This encourages the question of how the game can be set-up so that a solution exists; for additional examples of solvable configurations, see [13]. Before stating our theorem that addresses this question, we will need some additional terminology.

Let $S = \{s_1, s_2, \ldots, s_k\}$ and $T = \{t_1, t_2, \ldots, t_k\}$ be disjoint sets of distinct vertices. In this paper, a *paired disjoint k-path cover* of a graph G is a subgraph of G consisting of paths $P_1, P_2, \ldots, P_k$ such that the path P_i has endpoints s_i and t_i, and the vertex sets of the paths partition $V(G)$. We will often abbreviate the term "paired disjoint k-path cover" as "paired k-path cover" or simply "k-path cover". If G has a k-path cover for every choice of S and T, then G is said to be *paired k-to-k disjoint path coverable*.

Let G be a bipartite graph with partite sets V_1 and V_2. Let $|V_1| - |V_2| = \delta$. We say that $S \cup T$ is *balanced* if $|(S \cup T) \cap V_1| - |(S \cup T) \cap V_2| = 2\delta$. Note that the existence of a k-path cover of G for endpoints S and T implies $S \cup T$ is balanced. Hence, G is said to be *balanced paired k-to-k disjoint path coverable* if G has a k-path cover for every choice of S and T such that $S \cup T$ is balanced.

Let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ be an n-tuple of integers such that $d_i \geq 2$ for all i. The graph $Q_n^{\mathbf{d}}$ is a *hypertorus*, or an *n-dimensional torus*, which is defined on the same vertex set as $L_n^{\mathbf{d}}$ and has edges between $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ if and only if there exists a unique i, $1 \leq i \leq n$, such that $|u_i - v_i| = 1$ or $d_i - 1$, and for all $j \neq i$, $u_j = v_j$. A standard torus is simply $Q_2^{(d_1,d_2)}$.

We are now ready to state our main result.

Theorem 3.1. *Let $d_1 \geq 3$ and $d_2 \geq 3$ be integers. If d_1 and d_2 are not both even, then $Q_2^{(d_1,d_2)}$ is paired 2-to-2 disjoint path coverable. Otherwise, $Q_2^{(d_1,d_2)}$ is balanced paired 2-to-2 disjoint path coverable.*

In terms of the game Flow Free, Theorem 3.1 implies that given two pairs of endpoints on a d_1 by d_2 board with opposite sides identified, there exists a solution if and only if either d_1 and d_2 are not both even, or d_1 and d_2 are both even and the four endpoints are balanced, i.e., two endpoints are on white squares and the other two on black squares (when the board is colored like a standard chess board).

We now consider this game in higher dimensions on $Q_n^{\mathbf{d}}$. Our results are as follows. Theorem 2.2 addresses a game scenario in which for each pair, one dot is given and the other may be chosen by the player. Theorem 3.3 returns to the standard Flow Free scenario when played on a torus, but in higher dimensions.

Theorem 2.2. *Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k$ be k distinct vertices of $Q_n^{\mathbf{d}}$. Then there exist k distinct vertices $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_k$ of $Q_n^{\mathbf{d}}$, $\mathbf{f}_i \neq \mathbf{e}_j$ if $i \neq j$, such that $Q_n^{\mathbf{d}}$ can be partitioned into vertex-disjoint paths with endpoints $\mathbf{e}_i$ and $\mathbf{f}_i$. Note that if $\mathbf{e}_i = \mathbf{f}_i$, then the path is empty.*

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