ARTICLE IN PRESS

Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Sufficient conditions on the zeroth-order general Randić index for maximally edge-connected graphs

Zhibing Chen [a], Guifu Su [b,*], Lutz Volkmann [c]

[a] College of Mathematics and Statistics, Shenzhen University, Guangdong 518060, PR China

[b] School of Science, Beijing University of Chemical Technology, Beijing, 100029, PR China

[c] Lehrstuhl II fur Mathematik, RWTH Aachen University, 52056 Aachen, Germany

A R T I C L E   I N F O

Article history:
Received 6 November 2015
Received in revised form 14 August 2016
Accepted 3 November 2016
Available online xxxx

Keywords:
Edge-connectivity
Maximal edge-connected graphs
Zeroth-order general Randić index

A B S T R A C T

Let G be a connected graph with vertex set V, minimum degree δ and edge-connectivity λ. If α is a real number, then the zeroth-order general Randić index is defined by $\sum_{x\in V}\deg^{\alpha}(x)$, where $\deg(x)$ denotes the degree of the vertex x. A graph is maximally edge-connected if $\lambda=\delta$. In this paper, we present sufficient conditions for connected graphs (resp. connected triangle-free graphs) to be maximally edge-connected in terms of the zeroth-order general Randić index, the order and the minimum degree when $\alpha\in(-\infty,0)$ or $\alpha\in(1,2]$ (resp. $\alpha\in[-1,0)\cup(1,2]$).

1. Introduction

Let G be a finite and simple graph with vertex set $V=V(G)$ and edge set $E=E(G)$. Then the order and size of G are $n=|V|$ and $m=|E|$, respectively. The *degree* of a vertex $u\in V$ is the number of edges incident to u in G, denoted by $\deg(u)=\deg_G(u)$. The minimum value of these numbers is the *minimum degree* of G, denoted by $\delta=\delta(G)$. We write K_n for the complete graph of order n. An *edge-cut* of a connected graph G is a set of edges whose removal disconnects G. The edge connectivity $\lambda=\lambda(G)$ of a connected graph G is defined as the minimum cardinality of an edge-cut over all edge-cuts of G. An edge-cut S is a *minimum edge-cut* or a *λ-cut* if $|S|=\lambda(G)$. The inequality $\lambda(G)\le\delta(G)$ is immediate. We call a connected graph *maximally edge-connected*, if $\lambda(G)=\delta(G)$. Sufficient conditions for graphs to be maximally edge-connected were given by several authors, see for example the survey paper by Hellwig and Volkmann [11].

In this paper, we are concerned with the *zeroth-order general Randić index* which is defined for a connected graph G of order $n\ge 2$ by

$$R^0_\alpha(G)=\sum_{u\in V}\deg^{\alpha}(u),$$

where α is a real number. In 2005, Li and Zheng [15] proposed this index and named it the first general Zagreb index, we encourage the interested reader to consult [8,7] for more information. But nowadays, most authors refer to it as the zeroth-order general Randić index. At this point it is worth mentioning that R^0_2 and $R^0_{-0.5}$ correspond to the first Zagreb index, introduced by Gutman and co-workers [10,9], and the zeroth-order Randić index, defined by Kier and Hall [14], respectively. Some nice results on the zeroth-order general Randić index can be found, for example in [1,12,13]. The special case $\alpha=-1$

* Corresponding author.
E-mail addresses: chenzb@szu.edu.cn (Z. Chen), gfs1983@126.com (G. Su), volkm@math2.rwth-aachen.de (L. Volkmann).

http://dx.doi.org/10.1016/j.dam.2016.11.002
0166-218X/

Please cite this article in press as: Z. Chen, et al., Sufficient conditions on the zeroth-order general Randić index for maximally edge-connected graphs, Discrete Applied Mathematics (2016), http://dx.doi.org/10.1016/j.dam.2016.11.002

is the known *inverse degree* of graphs. The inverse degree first attracted attention through conjectures of the computer program Graffiti [6]. It has since been studied by several authors, see for example [3,5]. In [2], the authors present sufficient conditions for connected graphs to be maximally edge-connected in terms of the inverse degree, the order and the minimum degree.

Inspired by the results in [2], we give in this paper sufficient conditions for connected graphs (resp. connected triangle-free graphs) to be maximally edge-connected in terms of the zeroth-order general Randić index, the order and the minimum degree when $\alpha \in (-\infty, 0)$ or $\alpha \in (1, 2]$ (resp. $\alpha \in [-1, 0) \cup (1, 2]$). Examples will show that these conditions are best possible. Similar results for $\alpha \in (0, 1)$ and $\alpha \in (-\infty, -1]$ can be found in [17,18].

2. Preliminary lemmas

To obtain our main results, we first give some lemmas as necessary preliminaries. The first one is easy to prove and can be found in [16].

Lemma 2.1 ([16]). *If $x_1 - 2 \geq x_2 \geq 1$, then*
(i) $(x_1 - 1)^\alpha + (x_2 + 1)^\alpha < x_1^\alpha + x_2^\alpha$ *if* $\alpha \in (-\infty, 0)$ *or* $\alpha \in (1, +\infty)$ *;*
(ii) $(x_1 - 1)^\alpha + (x_2 + 1)^\alpha > x_1^\alpha + x_2^\alpha$ *if* $\alpha \in (0, 1)$.

Lemma 2.2. *Let $\alpha < 0$ be a real number, and let $x_1, x_2, \ldots, x_p$ and A be positive reals such that $\sum_{i=1}^{p} x_i \leq A$, then $\sum_{i=1}^{p} x_i^\alpha \geq p^{1-\alpha}A^\alpha$. If, in addition $x_1, x_2, \ldots, x_p, A$ are positive integers, and a, b are integers with $A = ap + b$ and $0 \leq b < p$, then $\sum_{i=1}^{p} x_i^\alpha \geq (p - b)a^\alpha + b(a + 1)^\alpha$.*

Proof. The first part follows by applying Jensen's Inequality. In the following we only give the proof of the rest part. Let $x_1, x_2, \ldots, x_p$ and A be positive integers with $\sum_{i=1}^{p} x_i \leq A$. We assume that $\{x'_1, x'_2, \ldots, x'_p\}$ are chosen such that $\sum_{i=1}^{p} x_i^\alpha$ is minimum. If none pair of elements in $\{x'_1, x'_2, \ldots, x'_p\}$ differ by more than 1, then $p - b$ of the x'_i equal a and the remaining b of the x'_i equal $a + 1$, and the inequality follows. So assume that two elements of $\{x'_1, x'_2, \ldots, x'_p\}$, say x'_1 and x'_2, differ by more than 1. Without loss of generality, we assume that $x'_1 > x'_2$. Let $y_1 = x'_1 - 1$, $y_2 = x'_2 + 1$ and $y_i = x'_i$ for $i \geq 3$. Then by Lemma 2.1(i), we have

$$\sum_{i=1}^{p} y_i^\alpha - \sum_{i=1}^{p} x_i^\alpha = (x'_1 - 1)^\alpha + (x'_2 + 1)^\alpha - (x'_1)^\alpha - (x'_2)^\alpha < 0,$$

which contradicts the choice of the $\{x'_1, x'_2, \ldots, x'_p\}$. This implies the desired result. ■

The next lemma follows from the definition of convex functions and can be found in [2].

Lemma 2.3 ([2]). *Let $\Phi(x)$ be a convex function on an interval $[L, R]$. If $l, r \in [L, R]$ and $l + r = L + R$, then $\Phi(L) + \Phi(R) \geq \Phi(l) + \Phi(r)$.*

Lemma 2.4 ([20]). *Let $x_1, x_2, \ldots, x_n \leq t$ be positive reals such that $\sum_{i=1}^{n} x_i = s \leq 2t$. If $f(x)$ is a strictly convex function on $(0, t)$, then $F(\frac{s}{n}, \frac{s}{n}, \ldots, \frac{s}{n}) \leq F(x_1, x_2, \ldots, x_n) \leq F(t, s - t, 0, \ldots, 0)$, where $F(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} f(x_i)$.*

A *complete r-partite graph*, denoted by $K_{n_1,n_2,\ldots,n_r}$, is a graph whose vertices can be partitioned into $r(\geq 2)$ sets so that each pair of vertices are connected by an edge if and only if they belong to different sets of the partition.

The *Turán graph* $T_{n,r}$ is the complete r-partite graph with b partite sets of size $a + 1$ and $r - b$ partite sets of size a, where $a = \lfloor \frac{n}{r} \rfloor$ and $b = n - ra$.

The following is a famous result due to Turán [19].

Lemma 2.5 ([19]). (i) *Among all the n-vertex graphs with no $(r + 1)$-clique, $T_{n,r}$ has the maximum number of edges.* (ii) $|E(T_{n,r})| \leq \left\lfloor \left(1 - \frac{1}{r}\right) \cdot \frac{n^2}{2} \right\rfloor$.

For two subsets X and Y of V, let $[X, Y]$ be the set of edges with one endpoint in X and the other one in Y, and $|[X, Y]|$ denotes the cardinality of $[X, Y]$.

The following lemma was proved by Dankelmann and Volkmann in [4].

Lemma 2.6 ([4]). *Let G be a connected graph. If there exist two disjoint, nonempty sets $X, Y \subset V(G)$ with $X \cup Y = V(G)$ and $|[X, Y]| < \delta$, then $|X| \geq \delta + 1$ and $|Y| \geq \delta + 1$.*

Lemma 2.7. *Let x and α be real numbers. Then*
(i) $x^\alpha + (x + \frac{1}{2})^\alpha \leq (x + 1)^\alpha + (x - \frac{1}{2})^\alpha$ *for* $\alpha \in (-\infty, 0]$ *or* $\alpha \in [1, +\infty)$.
(ii) $x^\alpha + (x + \frac{1}{2})^\alpha \geq (x + 1)^\alpha + (x - \frac{1}{2})^\alpha$ *for* $\alpha \in (0, 1)$.

Proof. Let $f(t) = t^\alpha - (t - \frac{1}{2})^\alpha$. It is easy to see that $f'(t) > 0$ for $\alpha \in (-\infty, 0]$ or $\alpha \in [1, +\infty)$, and so $f(t)$ is increasing. Hence, $f(x + 1) \geq f(x)$, consequently, $(x + 1)^\alpha - (x + \frac{1}{2})^\alpha \geq x^\alpha - (x - \frac{1}{2})^\alpha$. This completes the proof of (i). Since the proof of (ii) is analogous to that of (i), we omit the details here. ■