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Maximum Laplacian energy of unicyclic graphs

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A B S T R A C T

We study the Laplacian and the signless Laplacian energy of connected unicyclic graphs, obtaining a tight upper bound for this class of graphs. We also find the connected unicyclic graph on n vertices having largest (signless) Laplacian energy for $3 \leq n \leq 13$. For $n \geq 11$, we conjecture that the graph consisting of a triangle together with $n-3$ balanced distributed pendent vertices is the candidate having the maximum (signless) Laplacian energy among connected unicyclic graphs on n vertices. We prove this conjecture for many classes of graphs, depending on σ, the number of (signless) Laplacian eigenvalues bigger than or equal to 2.

1. Introduction and main results

For a simple graph $G = (V, E)$ with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, $|V(G)| = n$ and $|E(G)| = m$, the *Laplacian matrix* and the *signless Laplacian matrix* of G are $L(G) = \mathbf{D}(G) - \mathbf{A}(G)$, $Q(G) = \mathbf{D}(G) + \mathbf{A}(G)$, respectively, where $\mathbf{A}(G)$ is the $(0, 1)$-adjacency matrix of G and $\mathbf{D}(G)$ is the diagonal matrix of vertex degrees. The spectrum of $L(G)$ and $Q(G)$, composed by the non negative eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ and $q_1 \geq q_2 \geq \cdots \geq q_n \geq 0$ is called, respectively, the *Laplacian spectrum* and the *signless Laplacian spectrum* of G. When more than one graph is under consideration, we write $\mu_i(G)$ and $q_i(G)$ instead of μ_i and q_i.

The present paper is concerned with the well known parameters Laplacian and the signless Laplacian energy of G defined, respectively, as

$$LE = LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|, \tag{1}$$

$$SLE = SLE(G) = \sum_{i=1}^{n} \left| q_i - \frac{2m}{n} \right|. \tag{2}$$

We study the problem of determining the connected unicyclic graph with n vertices having largest (signless) Laplacian energy. In [6], it is proven that the star S_n has largest Laplacian energy among all trees on n vertices. Since the Laplacian spectrum and signless Laplacian spectrum of trees are the same, the result also holds for the signless Laplacian energy. Hence,

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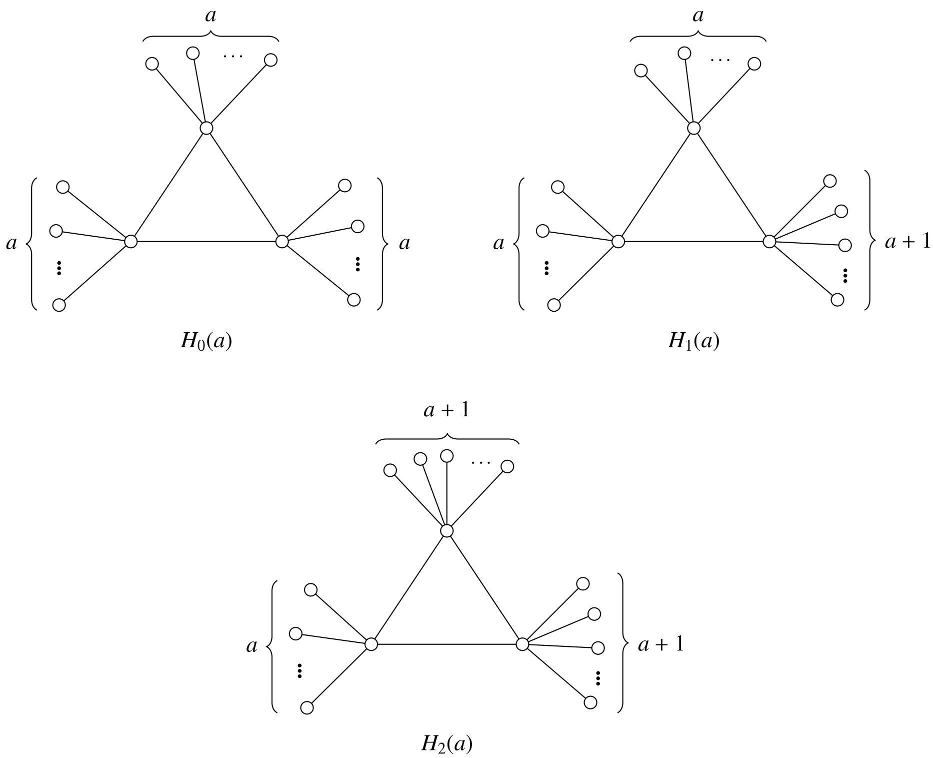

Fig. 1. Graphs $H_i(a)$, with $n = 3a + i + 3$ vertices.

a natural candidate for the connected unicyclic graph with largest (signless) Laplacian energy is S_n', the graph obtained from S_n by adding an edge, which is a minimum deviation from S_n.

It turns out that this is only true for small values of n. In fact we have the following conjecture.

Conjecture 1. *For $n \geq 11$, the connected unicyclic graph with n vertices having largest Laplacian and largest signless Laplacian energy is $H(n)$, the triangle together with $n - 3$ balanced distributed pendent vertices (see Fig. 1).*

By conducting an exhaustive[1] computation for all connected unicyclic graphs on $n < 14$ vertices we verify the following property.

Proposition 2. *Let G and G' be the unicyclic graphs having largest Laplacian and largest signless Laplacian energy, respectively. Then*

(1) $G = G' = C_n$ *for* $n = 3$;
(2) $G = G' = S_n'$ *for* $4 \leq n \leq 9$;
(3) $G = S_n'$ *and* $G' = H(10)$ *for* $n = 10$;
(4) $G = G' = H(n)$ *for* $n = 11, 12, 13$.

We also prove that Conjecture 1 is true for most graphs. More precisely, if G is a unicyclic graph of order $n \geq 12$ and σ is the number of Laplacian eigenvalues of G greater than or equal to 2, we prove that if $\sigma \geq 9$, then $H(n)$ has Laplacian energy larger than G. A similar result is stated in the final section for the signless Laplacian energy.

The main tool to prove this result is a very tight bound for this problem. We prove that for any connected unicyclic graph on $n \geq 4$ vertices, $LE(G)$ and $SLE(G)$ are bounded by $2n - \frac{4\sigma}{n}$.

We believe this bound is important on its own because not only it is a tool to prove our conjecture is true for many graphs, it also shows that the Laplacian energy of the candidate for largest Laplacian energy is within a factor of $O(1/n)$ of the bound.

We also shed some light on the parameter $\sigma(G) = \sigma (1 \leq \sigma \leq n - 1)$, the number of Laplacian eigenvalues greater than or equal to the average degree $\overline{d} = \frac{2m}{n}$. More precisely, σ is the largest positive integer such that

$$\mu_\sigma \geq \frac{2m}{n}. \tag{3}$$

[1] We use the software SAGE [13].

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