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On the algorithmic complexity of adjacent vertex closed distinguishing colorings number of graphs

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ABSTRACT

An assignment of numbers to the vertices of graph G is *closed distinguishing* if for any two adjacent vertices v and u the sum of labels of the vertices in the closed neighborhood of the vertex v differs from the sum of labels of the vertices in the closed neighborhood of the vertex u unless they have the same closed neighborhood (i.e. $N[u] = N[v]$). The *closed distinguishing number* of a graph G, denoted by $dis[G]$, is the smallest integer k such that there is a closed distinguishing labeling for G using integers from the set $\{1, 2, \ldots, k\}$. Also, for each vertex $v \in V(G)$, let $L(v)$ denote a list of natural numbers available at v. A *list closed distinguishing labeling* is a closed distinguishing labeling f such that $f(v) \in L(v)$ for each $v \in V(G)$. A graph G is said to be *closed distinguishing k-choosable* if every k-list assignment of natural numbers to the vertices of G permits a list closed distinguishing labeling of G. The closed distinguishing choice number of G, $dis_\ell[G]$, is the minimum natural number k such that G is closed distinguishing k-choosable. In this work we show that for each integer t there is a bipartite graph G such that $dis[G] > t$. This is an answer to a question raised by Axenovich et al. in (Axenovich et al., 2016) that how "dis" function depends on the chromatic number of a graph. It was shown that for every graph G with $\Delta \geq 2$, $dis[G] \leq dis_\ell[G] \leq \Delta^2 - \Delta + 1$ and also there are infinitely many values of Δ for which G might be chosen so that $dis[G] = \Delta^2 - \Delta + 1$ (Axenovich et al., 2016). In this work, we prove that the difference between $dis[G]$ and $dis_\ell[G]$ can be arbitrary large and show that for every positive integer t there is a graph G such that $dis_\ell[G] - dis[G] \geq t$. Also, we improve the current upper bound and give some number of upper bounds for the closed distinguishing choice number by using the Combinatorial Nullstellensatz. Among other results, we show that it is **NP**-complete to decide for a given planar subcubic graph G, whether $dis[G] = 2$. Also, we prove that for every $k \geq 3$, it is **NP**-complete to decide whether $dis[G] = k$ for a given graph G.

1. Introduction

In 2004, Karoński et al. in [19] introduced a new coloring of a graph which is generated via edge labeling. Let $f : E(G) \to \mathbb{N}$ be a labeling of the edges of a graph G by positive integers such that for every two adjacent vertices v and u, $S(v) \neq S(u)$, where $S(v)$ denotes the sum of labels of all edges incident with v. It was conjectured that three integer labels $\{1, 2, 3\}$ are sufficient for every connected graph, except K_2 [19] (1–2–3 Conjecture). Currently the best bound that was proved by

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Kalkowski et al. is five [18]. For more information we refer the reader to a survey on the 1–2–3 Conjecture and related problems by Seamone [27] (also see [4,6,8,10,12,24,30]). Different variations of distinguishing labelings of graphs have also been considered, see [5,7,17,20–22,25,26,28].

On the other hand, there are different types of labelings which consider the closed neighborhoods of vertices. In 2010, Esperet et al. in [13] introduced the notion of locally identifying coloring of a graph. A proper vertex-coloring of a graph G is said to be *locally identifying* if for any pair u, v of adjacent vertices with distinct closed neighborhoods, the sets of colors in the closed neighborhoods of u and v are different. In 2014, Aïder et al. in [1] introduced the notion of relaxed locally identifying coloring of graphs. A vertex-coloring of a graph G (not necessary proper) is said to be *relaxed locally identifying* if for any pair u, v of adjacent vertices with distinct closed neighborhoods, the sets of colors in the closed neighborhoods of u and v are different. Note that a relaxed locally identifying coloring of a graph that is similar to locally identifying coloring for which the coloring is not necessary proper. For more information see [14,16,27].

Motivated by the 1–2–3 Conjecture and the relaxed locally identifying coloring, the closed distinguishing labeling as a vertex version of the 1–2–3 Conjecture was introduced by Axenovich et al. [3]. For every vertex v of G, let $N[v]$ denote the closed neighborhood of v. An assignment of numbers to the vertices of a graph G is *closed distinguishing* if for any two adjacent vertices v and u the sum of labels of the vertices in the closed neighborhood of the vertex v differs from the sum of labels of the vertices in the closed neighborhood of the vertex u unless $N[u] = N[v]$ (i.e. they have the same closed neighborhood). The *closed distinguishing number* of a graph G, denoted by $dis[G]$, is the smallest integer k such that there is a closed distinguishing assignment for G using integers from the set $\{1, 2, \ldots, k\}$. For each vertex $v \in V(G)$, let $L(v)$ denote a list of natural numbers available at v. A *list closed distinguishing labeling* is a closed distinguishing labeling f such that $f(v) \in L(v)$ for each $v \in V(G)$. A graph G is said to be *closed distinguishing k-choosable* if every k-list assignment of natural numbers to the vertices of G permits a list closed distinguishing labeling of G. The *closed distinguishing choice number* of G, $dis_\ell[G]$, is the minimum natural number k such that G is closed distinguishing k-choosable. In this work we study closed distinguishing number and closed distinguishing choice number of graphs.

The closed distinguishing number of a graph G is the smallest integer k such that there is a closed distinguishing assignment for G using integers from the set $\{1, 2, \ldots, k\}$. In this work, we also consider another parameter, the minimum number of integers required in a closed distinguishing labeling. For a given graph G, the minimum number of integers required in a closed distinguishing labeling is called its *strong closed distinguishing number* $dis_s[G]$. Note that a vertex-coloring of a graph G (not necessary proper) is said to be strong closed distinguishing labeling if for any pair u, v of adjacent vertices with distinct closed neighborhoods, the multisets of colors in the closed neighborhoods of u and v are different.

2. Closed distinguishing labeling

In this section we study closed distinguishing number and closed distinguishing choice number of graphs.

2.1. The difference between $dis[G]$ and $dis_\ell[G]$

It was shown in [3] that for every graph G with $\Delta \geq 2$, $dis[G] \leq dis_\ell[G] \leq \Delta^2 - \Delta + 1$. Also, there are infinitely many values of Δ for which G might be chosen so that $dis[G] = \Delta^2 - \Delta + 1$ [3]. We prove that the difference between $dis[G]$ and $dis_\ell[G]$ can be arbitrary large and show that for every number t there is a graph G such that $dis_\ell[G] - dis[G] \geq t$.

Theorem 1. *For every positive integer t there is a graph G such that $dis_\ell[G] - dis[G] \geq t$.*

2.2. The complexity of determining $dis[G]$

Let $T \neq K_2$ be a tree. It was shown [3] that $dis_\ell[T] \leq 3$ and $dis[T] \leq 2$. Here, we investigate the computational complexity of determining $dis[G]$ for planar subcubic graphs and bipartite subcubic graphs.

Theorem 2. *For a given planar subcubic graph G, it is* **NP***-complete to decide whether $dis[G] = 2$.*

Although for a given tree T, we can compute $dis[T]$ in polynomial time [3], but the problem of determining the closed distinguishing number is hard for bipartite graphs.

Theorem 3. *For a given bipartite subcubic graph G, it is* **NP***-complete to decide whether $dis[G] = 2$.*

Note that in the proof of Theorem 3, we reduced Not-All-Equal to our problem and the planar version of Not-All-Equal is in **P** [23], so the computational complexity of deciding whether $dis[G] = 2$ for planar bipartite graphs remains unsolved.

Theorem 4. *For every integer $t \geq 3$, it is* ***NP****-complete to decide whether $dis[G] = t$ for a given graph G.*