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Note

An infinite family of 2-connected graphs that have reliability factorisations

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ABSTRACT

The reliability polynomial $\Pi(G, p)$ gives the probability that a graph is connected given each edge may fail independently with probability $1 - p$. Two graphs are *reliability equivalent* if they have the same reliability polynomial. It is well-known that the reliability polynomial can factorise into the reliability polynomials of the blocks of a graph. We give an infinite family of graphs that have no cutvertex but factorise into reliability polynomials of graphs of smaller order.

Brown and Colbourn commented that it was not known if there exist pairs of reliability equivalent graphs with different chromatic numbers. We show that there are infinitely many pairs of reliability equivalent graphs where one graph in each pair has chromatic number 3 and the other graph has chromatic number 4.

1. Introduction

The *reliability polynomial* $\Pi(G, p)$ of a graph $G = (V, E)$ gives the probability that a graph is connected given each edge can fail independently with probability $1 - p$. This polynomial is a specialisation of the Tutte polynomial, a two-variable polynomial that plays an important role in many enumeration problems on graphs [11,12,6]. Other specialisations of the Tutte polynomial include the chromatic polynomial that counts the number of proper colourings of a graph and the flow polynomial that counts network flows [11,1,13].

Graphs G and H are *reliability (chromatically) equivalent* if they have the same reliability (chromatic) polynomial. If G and H have the same Tutte polynomial, then clearly they are both reliability equivalent and chromatically equivalent. However, there exist pairs of graphs that are reliability equivalent but are not chromatically equivalent, and pairs of graphs that are chromatically equivalent but not reliability equivalent [2]. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number of colours required to properly colour G (and also the smallest natural number that is not a root of the chromatic polynomial of G). Brown and Colbourn comment in [2] that they know of no examples of graphs that are reliability equivalent but have different chromatic numbers. In this article we give infinitely many examples of pairs of graphs that are reliability equivalent but have different chromatic numbers. Here graphs can have multiple edges and loops.

The reliability polynomial can be computed using the following recurrence:

$$\Pi(G, p) = \begin{cases} p\Pi(G/uv, p) & \text{if } uv \text{ is a bridge,} & \textbf{(C1)} \\ \Pi(G \setminus uv, p) & \text{if } uv \text{ is a loop,} & \textbf{(C2)} \\ \Pi(G, p) = p\Pi(G/uv, p) + (1-p)\Pi(G \setminus uv) & \text{otherwise} & \textbf{(C3)} \end{cases}$$

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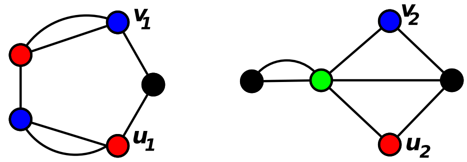

(a) G. (b) J.

Fig. 1. Graphs G and J.

where $G \setminus uv$ is the graph obtained by deleting the edge uv in G and G/uv is the contracted graph which is obtained by first removing the edge uv and then identifying the vertices u and v in G. Any multiple edges and loops obtained in this process are retained. We use the notation K_n to denote the simple complete graph of order n. When $G = K_1$, we have

$$\Pi(K_1, p) = 1. \quad \textbf{(C4)}$$

A *cutvertex* is a vertex whose removal increases the number of components of the graph. A *block* is a maximal connected subgraph that has no cutvertex. A graph G is *k-connected* if G cannot be disconnected by removing any set of $<k$ vertices in the graph. Let $G_1 \bullet G_2$ denote a graph obtained by identifying a vertex in G_1 with a vertex in G_2. As the reliability polynomial is a partial evaluation of the Tutte polynomial [13], it is multiplicative over blocks [11]. In particular, if a graph has a cutvertex, then

$$\Pi(G_1 \bullet G_2, p) = \Pi(G_1, p)\Pi(G_2, p). \quad \textbf{(C5)}$$

More generally, we say a graph G has a *reliability factorisation* if there exist graphs G_1 and G_2 that are not isomorphic to K_1 such that

$$\Pi(G, p) = \Pi(G_1, p)\Pi(G_2, p).$$

We say that G_1 and G_2 are the *reliability factors* of G. Any pair of 2-isomorphic graphs have the same reliability polynomial [14] which gives us the following property:

$$\Pi(G_1(u_1, v_1) \cup_2 G_2(u_2, v_2)) = \Pi(G_1(v_1, u_1) \cup_2 G_2(u_2, v_2)) \quad \textbf{(C6)}$$

where $G_1(u_1, v_1) \cup_2 G_2(u_2, v_2)$ is the graph obtained from G_1 and G_2 by identifying u_1 and u_2 as one vertex and v_1 and v_2 as another vertex.

It is clear that any graph that has a cutvertex has a reliability factorisation. It was shown in [5] that there are 2-connected graphs that have reliability factorisations, and all cases with up to 13 edges were identified. In this article, we give an infinite family of 2-connected graphs that have reliability factorisations. These graphs have a two-vertex cut, that is, a set of two vertices whose removal disconnects the graph. As any graph that has a one-vertex cut (i.e., a cutvertex) has a reliability factorisation, it is interesting to identify cases where the minimum vertex cut is greater than one, but the graph still has a reliability factorisation. We say that a graph has a *strong reliability factorisation* if it has a reliability factorisation but has no cutvertex. A reliability polynomial is said to have *a strong reliability factorisation* if it is the reliability polynomial of a graph that has a strong reliability factorisation.

A computer search of reliability polynomials of all connected graphs with at most 13 edges identified 515 reliability polynomials of graphs that have strong reliability factorisations [5]. More than 22% of these reliability polynomials are the reliability polynomials of graphs belonging to our infinite family.

Furthermore, we construct an infinite family of pairs of graphs that have the same reliability polynomial but different chromatic numbers. This addresses the comment by Brown and Colbourn in [2] that there were no known examples of this kind.

2. An infinite family

In [8] we introduced the notion of a *certificate* to prove properties of the chromatic polynomial. A certificate is a sequence of expressions $(E_0, E_1, \ldots, E_k)$ where each expression E_i, $i \in [1, k]$, is obtained from E_{i-1} by a *certificate step* that uses a property of the graph polynomial or an algebraic property. Certificates have been used to prove algebraic properties of a number of different graph polynomials including the chromatic polynomial [8,3,4,9,10], the stability polynomial [7] and the reliability polynomial [5]. In this article, we use certificates to show that our infinite family of graphs that have strong reliability factorisations.

We say $G = G_1 \cup_{u,v} G_2$ where G is the graph obtained by identifying vertices u_1 and v_1 in graph G_1 with vertices u_2 and v_2 in G_2 respectively. The identified vertices are labelled u and v respectively.

Theorem 1. *Let G be the graph in Fig. 1(a). The graph $G \cup_{u,v} H$ has a reliability factorisation for any graph H.*

Proof. Fig. 3 gives a certificate of reliability factorisation for the graph $G \cup_{u,v} H$ based on the properties **(C1)**–**(C6)** where G is the graph in Fig. 1(a) and the reliability factors are the graphs C_2 and A in Fig. 2. □