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The splitting technique in monotone recognition

L. Aslanyan *, H. Sahakyan
Institute for Informatics and Automation Problems, National Academy of Sciences, P. Sevak str. 1, 0014 Yerevan, Armenia

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Dedicated to the bright memory of Levon Khachatrian

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ABSTRACT

We consider the problem of query based algorithmic identification/recognition of monotone Boolean functions, as well as of binary functions defined on multi-valued discrete grids. Hansel's chain-split technique of n-cubes is a well known effective tool of monotone Boolean recognition. An extension by Alekseev is already applied to the grid case. The practical monotone recognition on n-cubes is provided by the so called chain-computation algorithms that is not extended to the case of multi-valued grids. We propose a novel split construction based on partitioning the grid into sub-grids and into discrete structures that are isomorphic to binary cubes. Monotonicity in a multi-valued grid implies monotonicity in all induced binary cubes and in multi-valued sub-grids. Applying Hansel's technique for identification of monotone Boolean functions on all appearing binary cubes, and Alekseev's algorithm on all sub-grids leads to different scenarios of reconstruction of monotone functions. On one hand such partitioning technique makes parallel recognition possible, on the other hand — the method can be used in practical identification algorithms due to simple structures and easily calculable quantities appearing after the partition to the n-cubes. Complexity issues of considered algorithms were also elaborated.

1. Introduction and preliminaries

Monotone Boolean functions appear in various ICT applications, such as design of electronic schemes, pattern recognition, discrete optimization, cryptography and others [2,6,21]. Monotone Boolean functions are studied from different viewpoints and they are known as a type of high complexity objects [13,11,14,15,18–20]. Often researchers link this complexity to the Sperner families of partially ordered sets of elements [8]. As a rule, the problem is considered in specific posets — such as the binary cube, and the multi-valued multidimensional grid. A number of results in the domain of structural optimization of monotone Boolean functions and their recognition are obtained by G. Hansel, V. Korobkov, A. Korshunov, G. Tonoyan, N. Zolotykh, V. Alekseev, A. Serjantov and others [11,1,14–16,23,27,24–26].

The exact recognition algorithm, optimal for the n-cube and in the sense of the Shannon complexity criterion, is given by G. Hansel in [11]. The algorithm is based on partitions of the n-dimensional binary cube into disjoint chains, that is effective and very much transparent and understandable. A direct generalization of Hansel's approach to the multi-valued case is obtained by V. Alekseev in [1]. For one particular sub-case this result is improved in [23].

As an example consider an applied problem that can use monotone recognition.

Assume we are given a set of m linear inequalities, $A \cdot X \leq B$ on the set of variables $X = (x_1, x_2, \ldots, x_n)$. In general, this system can be inconsistent. The problem is to find algorithmically one or all maximal consistent subsets of inequalities. If to observe that a subset of a consistent set of inequalities is consistent, and if to code the involvement of individual inequalities

* Corresponding author.
E-mail addresses: lasl@sci.am (L. Aslanyan), hasmik@ipia.sci.am (H. Sahakyan).

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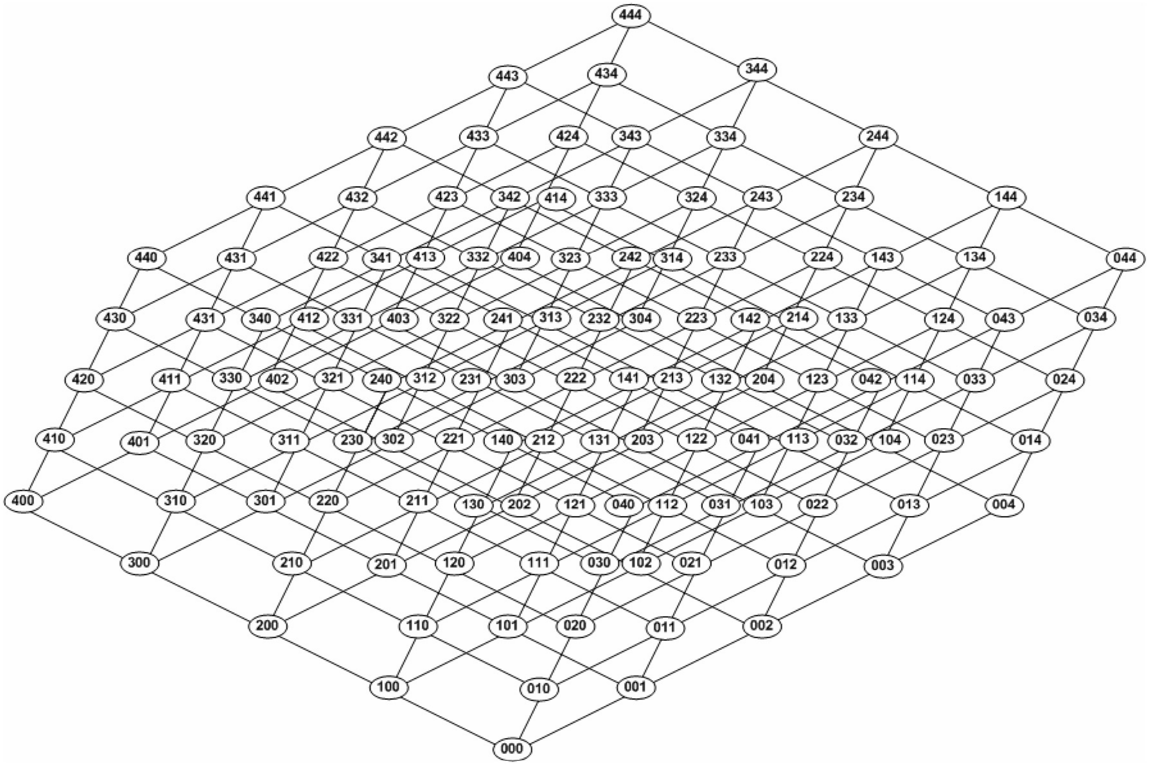

Fig. 1. Hasse diagram of Ξ_5^3, circles correspond to vertices.

into the subsystem under consideration by binary vectors, we get a monotone Boolean function, such that zero values of the function represent all consistent subsets of the system of inequalities. The maximal consistent sets correspond to the upper zeros of the monotone Boolean function.

Other typical applications appear in data mining area [22], where frequent sets of items compose monotone Boolean functions. This is for the case of market basket analysis. If to take into consideration the item quantities in the basket, then we obtain the same problem on multi-valued grids.

In this paper we consider a novel algorithmic resource for elaboration and identification of monotone functions defined on multi-valued grids. We propose a principally new approach based on partitioning of grids into non-intersecting discrete structures, that are isomorphic to binary cubes. Monotonicity of a function in the multi-valued grid implies monotonicity in all induced binary cubes. Monotonicity is retained also in induced multi-valued sub-grids. Hence, applying Hansel's method (and its extensions) for identification of monotone functions in all induced binary cubes and in sub-grids, and then integrating the results, leads to an alternative way of reconstruction of monotone functions defined on the multi-valued grids. The method can be used in practical algorithms of identification due to simple structures and easily calculable quantities in the n-cubes. In a general characterization, the new approach provides a binary cube partition technique vs. the chain partition technique used so far.

Let $\Xi_{m+1} = \{0, 1, \dots, m\}$ and Ξ_{m+1}^n denote the set of vertices of the n-dimensional $(m+1)$-valued discrete grid defined as:

$$\Xi_{m+1}^n = \{(a_1, \dots, a_n) : a_i \in \Xi_{m+1} \text{ for all } i \in \overline{1, n} = \{1, 2, \dots, n\}\}.$$

We place a component-wise partial order on $\Xi_{m+1}^n : (a_1, \dots, a_n) \le (b_1, \dots, b_n)$ if and only if $a_i \le b_i$ for all $i \in \overline{1, n}$; and define the rank of an element $(a_1, \dots, a_n)$ as: $a_1 + \dots + a_n$. Then, Ξ_{m+1}^n is a ranked partially ordered set. Consider the geometric representation of Ξ_{m+1}^n through the Hasse diagram. The diagram has $m \cdot n + 1$ levels, numbered from 0 (lower level) to $m \cdot n$; the kth level contains all vertices at rank k. Edges connect those vertices in neighbor levels related by a cover relation. Let us demonstrate the Hasse diagram of Ξ_5^3 (see Fig. 1).

Consider a binary function f: $\Xi_{m+1}^n \to \{0, 1\}$. We say that f is *monotone* if for any two vertices $a, b \in \Xi_{m+1}^n$, if $a > b$ then $f(a) \ge f(b)$. For $m = 1$ we get monotone Boolean functions defined on the n-dimensional unit cube $E^n = \{(x_1, \dots, x_n) : x_i \in \{0, 1\}$ *for all* $i \in \overline{1, n}\}$.

$a^1 \in \Xi_{m+1}^n$ is a *lower unit* of some monotone function f if $f(a^1) = 1$, and $f(a) = 0$ for every $a \in \Xi_{m+1}^n$, which is less than a^1. $a^0 \in \Xi_{m+1}^n$ is an *upper zero* of monotone function f if $f(a^0) = 0$, and $f(a) = 1$ for every $a \in \Xi_{m+1}^n$, which is greater than a^0.

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