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An intersection theorem for systems of finite sets

Christian Bey

Universität Osnabrück, Institut für Mathematik, 49069 Osnabrück, Germany

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ABSTRACT

For nonnegative reals ω, ψ and natural $t \le k \le (n+t-1)/2$, the maximum of

$$\omega \left| \mathcal{A} \cap \binom{[n]}{k} \right| + \psi \left| \mathcal{A} \cap \binom{[n]}{n+t-1-k} \right|$$

among all t-intersecting set systems $\mathcal{A} \subseteq 2^{[n]}$ is determined.

1. Introduction and result

Throughout this note, n, k, t are positive integers with $n \ge k \ge t$. Let $[n]$ denote the set $\{1, 2, \ldots, n\}$, $2^{[n]}$ the powerset of $[n]$, and $\binom{[n]}{k}$ the set of all k-element subsets of $[n]$. For every $\mathcal{A} \subseteq 2^{[n]}$ and nonnegative integer i we set $\mathcal{A}_i := \mathcal{A} \cap \binom{[n]}{i}$.

A set system $\mathcal{A} \subseteq 2^{[n]}$ is called t-intersecting if $|A_1 \cap A_2| \ge t$ for all $A_1, A_2 \in \mathcal{A}$. Set

$$I(n,t) := \{\mathcal{A} \subseteq 2^{[n]} : \mathcal{A} \text{ is } t\text{-intersecting}\}.$$

For every nonnegative integer r, let

$$\mathcal{B}(r) := \{A \subseteq [n] : |A \cap [t+2r]| \ge t + r\}.$$

Obviously, $\mathcal{B}(r) \in I(n,t)$.

We recall the following two basic results in extremal set theory.

Theorem 1 (*Katona [5]*).

(i) *Let $\mathcal{A} \in I(n,t)$, $t \le k \le \frac{n+t-1}{2}$ and $0 \le \omega \le 1 + \frac{t-1}{k-t+1}$. Then*

$$\omega|\mathcal{A}_k| + |\mathcal{A}_{n+t-1-k}| \le \binom{n}{n+t-1-k}.$$

Equality holds in case of $2 \le t \le k < \frac{n+t-1}{2}$ iff $\mathcal{A}_k \cup \mathcal{A}_{n+t-1-k}$ is isomorphic to

- $\mathcal{B}_{n+t-1-k}(k-t+1) = \binom{[n]}{n+t-1-k}$ *if* $\omega < 1 + \frac{t-1}{k-t+1}$
- $\mathcal{B}_{n+t-1-k}(k-t+1)$ *or* $\mathcal{B}_k(k-t) \cup \mathcal{B}_{n+t-1-k}(k-t)$ *if* $\omega = 1 + \frac{t-1}{k-t+1}$,

and in case of $2 \le t \le k = \frac{n+t-1}{2}$ iff $\omega = 1 + \frac{t-1}{k-t+1}$ and $\mathcal{A}_{\frac{n+t-1}{2}}$ is isomorphic to $\mathcal{B}_{\frac{n+t-1}{2}}(k-t) = \binom{[2k-t]}{k}$.

E-mail address: christian.bey@uni-osnabrueck.de.

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(ii) *Let* $\mathcal{A} \in I(n,t)$. *Then*

$$|\mathcal{A}| \le \left|\mathcal{B}\left(\left\lfloor \frac{n-t}{2} \right\rfloor\right)\right| = \begin{cases} \displaystyle\sum_{k=\frac{n+t}{2}}^{n} \binom{n}{k} & \text{if } 2 \mid (n+t) \\ \displaystyle 2\sum_{k=\frac{n-1+t}{2}}^{n-1} \binom{n-1}{k} & \text{if } 2 \nmid (n+t). \end{cases}$$

Equality holds in case of $t \ge 2$ *iff* $\mathcal{A}$ *is isomorphic to* $\mathcal{B}\left(\lfloor \frac{n-t}{2} \rfloor\right)$.

Note that (ii) follows from the case $\omega = 1$ in (i) by adding over k all stated inequalities. Note also that the case $t = 1$ is easily dealt with by considering complements.

Theorem 2 (*Ahlswede, Khachatrian [1]*). *Let* $\mathcal{A} \in I(n,t)$, $t \le k \le \frac{n+t-1}{2}$. *Let* $r \in [0, k-t]$ *be the smallest integer satisfying*

$$\left(2 + \frac{t-1}{r+1}\right)(k-t+1) \le n. \tag{1}$$

Then

$$|\mathcal{A}_k| \le |\mathcal{B}_k(r)|.$$

Equality holds in case of $(k,t) \ne (\frac{n}{2}, 1)$ *iff* $\mathcal{A}_k$ *is isomorphic to*

- $\mathcal{B}_k(r)$ *if strict inequality holds in* (1)
- $\mathcal{B}_k(r)$ *or* $\mathcal{B}_k(r+1)$ *if equality holds in* (1).

The cases $t = 1$ and n sufficiently large are covered by the classical Erdős–Ko–Rado Theorem [4]. See the first section of [1] for further celebrated previously obtained partial results of Theorem 2.

Our contribution in this note is the following intersection theorem, which can be seen as a common extension of Theorem 1(i) and Theorem 2.

Theorem 3. *Let* $\mathcal{A} \in I(n,t)$, $t \le k \le \frac{n+t-1}{2}$ *and* $\omega, \psi \ge 0$ *(not both 0). Then*

$$\omega\,|\mathcal{A}_k| + \psi\,|\mathcal{A}_{n+t-1-k}| \le \max\{\omega\,|\mathcal{B}_k(r)| + \psi\,|\mathcal{B}_{n+t-1-k}(r)| \,:\, 0 \le r \le k-t+1\}.$$

Equality holds in case of $t \ge 2$ *iff* $\mathcal{A}_k \cup \mathcal{A}_{n+t-1-k}$ *is isomorphic to one of the systems* $\mathcal{B}_k(r) \cup \mathcal{B}_{n+t-1-k}(r)$ *which attains the maximum.*

The case $\psi = 0$ is covered by Theorem 2, and the case $\omega/\psi \le 1 + \frac{t-1}{k-t+1}$ is covered by Theorem 1(i). In the case $\psi \ne 0$ (w.l.o.g. $\psi = 1$) the following more precise theorem holds.

Theorem 3′. *Let* $\mathcal{A} \in I(n,t)$, $t \le k < \frac{n+t-1}{2}$ *and* $\omega \ge 0$. *Let* $r \in [0, k-t]$ *be the smallest integer satisfying*

$$(1+\omega)\left(n - \left(2 + \frac{t-1}{r+1}\right)(k-t+1)\right) \ge \left(2 + \frac{t-1}{r+1}\right)(n-2k+t-1) \tag{2}$$

or, in case of (2) *does not hold for* $r = k-t$, *let* $r := k-t+1$. *Then*

$$\omega\,|\mathcal{A}_k| + |\mathcal{A}_{n+t-1-k}| \le \omega\,|\mathcal{B}_k(r)| + |\mathcal{B}_{n+t-1-k}(r)|.$$

Equality holds in case of $t \ge 2$ *iff* $\mathcal{A}_k \cup \mathcal{A}_{n+t-1-k}$ *is isomorphic to*

- $\mathcal{B}_k(r) \cup \mathcal{B}_{n+t-1-k}(r)$ *if strict inequality holds in* (2)
- $\mathcal{B}_k(r) \cup \mathcal{B}_{n+t-1-k}(r)$ *or* $\mathcal{B}_k(r+1) \cup \mathcal{B}_{n+t-1-k}(r+1)$ *if equality holds in* (2)
- $\mathcal{B}_k(k-t+1) \cup \mathcal{B}_{n+t-1-k}(k-t+1) = \binom{[n]}{n+t-1-k}$ *if* $r = k-t+1$.

Note that for $r = k-t$ the inequality (2) reduces to $\omega \ge 1 + \frac{t-1}{k-t+1}$.

Theorem 3′ has implications for estimations of shadows of intersecting set systems. This will be explored elsewhere.

The proof of Theorem 3′ uses the powerful methods developed by Ahlswede and Khachatrian in [1,2]. We will not prove the uniqueness statement in this note.

2. Proof

The case $t = 1$ is easily dealt with by using complements and applying the Erdős–Ko–Rado Theorem. Also, as in Theorem 1(i), it suffices to consider the case $\omega \ge 1$.