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A strong connectivity property of the generalized exchanged hypercube

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A R T I C L E I N F O

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Dedicated to the memory of Levon H. Khachatrian

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A B S T R A C T

We show that, when a linear number of vertices are removed from a generalized exchanged hypercube, its surviving graph consists of a large connected component and smaller component(s) containing altogether a rather limited number of vertices. This result can be applied to obtain a number of fault tolerant properties of this interesting structure.

1. Introduction

Besides having made great accomplishments in the area of extremal combinatorics, e.g., [1], Prof. Khachatrian and his colleagues have also made important contributions to the study of interconnection networks, including [3,2]. In particular, analysis and construction of minimum broadcast networks were made in [2], where fault tolerance plays a crucial part, just like in this paper. We cannot help but observe that the research area of Interconnection networks started to gain traction in the late 1900s. If Prof. Khachatrian did not pass away on January 30, 2002, he would have made many more contributions in this area.

Thanks to a rapid and consistent technical progress in networking hardware, multi-processor systems have become a reality, where an interprocessor communication enabling network plays a crucial role. It is certainly unavoidable that some of the processing nodes and/or links within such a system will fault. To have an effective system to work with, we are naturally interested in the fault tolerant properties of these structures, when we seek answers to such questions as how many faulty nodes will disrupt such a structure, or disconnect its associated graph in graph theory terms; how disrupted the surviving structure (graph) will become when a certain number of nodes become faulty, thus effectively removed; and even how could we know exactly which ones are faulty so that connectivity of the system can be restored?

Answers to most of these questions are often expressed in terms of connectivity related properties of a graph underlying such a surviving structure [4,13,16,22–24]. As a result, we know that some of these graphs are robust in the sense that they

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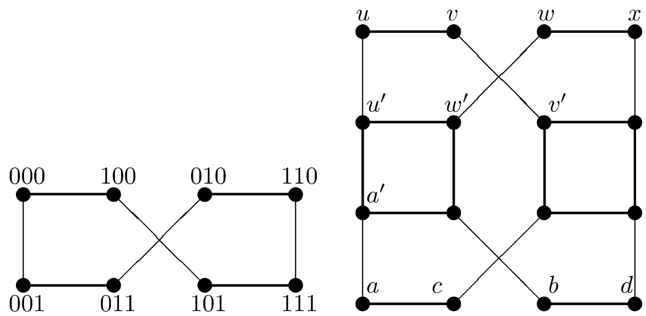

(a) $EH(1, 1)$. (b) A $GEH(1, 2)$.

Fig. 1. Simple exchanged hypercubes.

will not be disconnected unless all the neighbors of a vertex are removed (maximum connectivity) [4]; and even a structure is disconnected, if not too many vertices are removed, it remains largely intact in the sense that the surviving graph consists of a large connected component and some other components with a small number of vertices altogether, e.g., at most one vertex (tight super-connectivity) [4], each component contains at least two vertices (super connectivity) [11,10,18,21], or at most two vertices altogether (strong connectivity), [4,5,7,14]. In particular, some general and precise results to this regard have been obtained for the hypercube to address its super connectivity and component connectivity [13,21–24], extending the normal vertex connectivity of a non-complete graph, as commonly used in forming interconnection networks.

In this paper, we further extend these results to the class of generalized exchanged hypercubes, whose definition is forthcoming in the next section. In particular, we demonstrate that, after removing a linear number of vertices from a generalized exchanged hypercube, the surviving graph is either connected or consists of a large connected component and small components containing a small number of vertices. Such a general connectivity result can be applied to derive several fault-tolerance related network parameters such as the restricted connectivity [18,21], the cyclic vertex-connectivity [8,25], the component connectivity [13,22–24], and the conditional diagnosability [7,12,14] of this class of generalized exchanged hypercubes.

The rest of this paper proceeds as follows: We review the concept of the exchanged hypercube as proposed in [15] and some of its basic properties, propose the class of generalized exchanged hypercubes in Section 2, and obtain several basic connectivity results for this latter general structure in Section 3. We then, as the main result of this paper, present the promised connectivity property of the generalized exchanged hypercube in Section 4, and conclude this paper with some final remarks in Section 5.

2. The exchanged hypercube and its generalization

The n-dimensional hypercube, often referred to as the n-cube and denoted by Q_n, is perhaps one of the most studied and utilized interconnection structures. Several hypercube variants have since been suggested, including augmented cubes, crossed cubes, enhanced cubes, möbius cubes, and twisted cubes. The exchanged hypercube was proposed in [6,15] as another variant of the hypercube, where about half of the edges are systematically removed [6, Theorem 2]. Besides addressing a scaling issue as associated with the hypercube structure, the exchanged hypercube also manages to inherit several desirable properties of the hypercube such as incremental expandability [6], bipancyclicity [17], connectivity and super connectivity [18], and existence of a fault-tolerant routing algorithm [15]. We will further study some of its connectivity properties in this paper.

The exchanged hypercube, denoted by $EH(s, t), s, t \geq 1$, is defined as an undirected graph (V, E), where V is the collection of all the binary strings of length $s + t + 1$. Hence, $|V(EH(s, t))| = 2^{s+t+1}$. For $u \in V(EH(s, t))$, we denote u by $A(u)B(u)C(u)$, where $A(u) = a_{s-1} \cdots a_0$, $B(u) = b_{t-1} \cdots b_0$, and $C(u) = c$, sometimes referred to as the C *bit* of u henceforth. Let $u, v \in V(EH(s, t))$, $(u, v) \in E$ if and only if it falls into one of the following three mutually exclusive cases: E_1: $C(u) \neq C(v)$, but $A(u) = A(v)$ and $B(u) = B(v)$; E_2: $C(u) = C(v) = 0$, $A(u)$ and $A(v)$ differ in exactly one bit in position $p \in [0, s)$, while $B(u) = B(v)$; and E_3: $C(u) = C(v) = 1$, $A(u) = A(v)$, but $B(u)$ and $B(v)$ differ in exactly one bit in position $p \in [0, t)$.

Fig. 1(a) shows an example of exchanged hypercube $EH(1, 1)$, where (000, 001), (000, 100), and (001, 011) are examples of E_1, E_2, and E_3 edges, respectively.

Each collection of 2^s vertices u, where $C(u) = 0$ and whose B segments are identical, forms a Q_s, via the E_2 edges. Clearly, there are a total of 2^t such hypercubes in $EH(s, t)$, each referred to as a *Class*-0 *cluster*. Similarly, each collection of 2^t vertices u, where $C(u) = 1$ and whose A segments are identical, forms a Q_t, via E_3 edges. There are also a total of 2^s such hypercubes in $EH(s, t)$, each a *Class*-1 *cluster*. We thus refer to both E_2 and E_3 edges collectively as *cube edges*. Moreover, each vertex u in a cluster is adjacent to a unique vertex in a cluster of opposite class via an E_1 edge, denoted by u' in the rest of this paper. By definition, $A(u)B(u) = A(u')B(u')$ but $C(u') = \overline{C(u)}$, namely, the complement of the C *bit* of u. Since these E_1 edges pair vertices belonging to different clusters, they are referred to as the *cross edges*.