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On the maximum number of edges in a hypergraph with given matching number

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ABSTRACT

The aim of the present paper is to prove that the maximum number of edges in a 3-uniform hypergraph on n vertices and matching number s is

$$\max\left\{\binom{3s+2}{3}, \binom{n}{3} - \binom{n-s}{3}\right\}$$

for all $n, s,\ n \geq 3s+2$.

1. Introduction

Let $[n] = \{1, 2, \ldots, n\}$ be a finite set and $\mathcal{F} \subset \binom{[n]}{k}$ a k-uniform hypergraph. The matching number $\nu(\mathcal{F})$ is the maximum number of pairwise disjoint edges in $\mathcal{F}$. One of the classical problems in extremal set theory is to determine the maximum number of edges in a k-uniform hypergraph with matching number 1. This was solved by Erdős, Ko and Rado [5], who proved that for $n \geq 2k$ this maximum is $\binom{n-1}{k-1}$. There are two natural ways to generalize this problem for k-uniform hypergraphs. One is to consider the maximum number of edges for matching number 2, 3, etc. This is the problem that we shall solve in this paper for $k = 3$. The other one is to make the restriction "matching number is one" stronger by requiring that any two edges intersect in at least t elements (t is a fixed integer, $k > t > 1$). Such a family is called *t-intersecting*. Let us consider the following construction.

$$\mathcal{F}(n, k, t, i) = \{F \subset [n] :\ |F| = k, |F \cap [t+2i]| \geq t+i\}.$$

In 1976 the author [6] made the following conjecture. For all n, k and t, such that $n > 2k - t$, and for every k-uniform t-intersecting hypergraph $\mathcal{F}$ on n vertices one has

$$|\mathcal{F}| \leq \max_i |\mathcal{F}(n, k, t, i)|.$$

In 1987 Füredi and the author [11] showed that for every i in the range the conjecture is true for all pairs n and k if $t > t(i)$, that is $|\mathcal{F}(n, k, t, i)|$ is the largest. However, it was not until ten years later that Ahlswede and Khachatrian [1] succeeded in proving the conjecture completely.

Fixing the matching number, say s, there are two very natural constructions for k-graphs with that matching number:

$$\mathcal{A}_k = \binom{[ks+k-1]}{k}, \quad \text{and}$$

$$\mathcal{A}_1(n) = \left\{F \in \binom{[n]}{k} : F \cap [s] \neq \emptyset\right\}.$$

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In 1965 Paul Erdős made the following.

Conjecture 1.1 (*Matching Conjecture* [3]). *If $\mathcal{F} \subset \binom{[n]}{k}$ satisfies $\nu(\mathcal{F}) = s$ then*

$$|\mathcal{F}| \le \max\{|\mathcal{A}_1(n)|, |\mathcal{A}_k|\}.$$

In the same paper Erdős proved the conjecture for $n > n_0(k, s)$. Let us mention that the conjecture is trivial for $k = 1$, and it was proved for graphs ($k = 2$) by Erdős and Gallai [4].

There were several improvements on the bound $n_0(k, s)$. Bollobás, Daykin and Erdős [2] proved $n_0(k, s) \le 2k^3 s$ and recently Huang, Loh and Sudakov [13] improved it to $n_0(k, s) \le 3k^2 s$. The current record is due to the present author [9], it is $n_0(k, s) \leqq (2s + 1)k - s$.

The aim of the present paper is to prove

Theorem 1.1. *The conjecture is true for $k = 3$.*

We should mention that our proof relies partly on ideas from Frankl–Rödl–Ruciński [12], who proved $n_0(3, s) \le 4(s+1)$ and the recent result of Łuczak and Mieczkowska [15] who proved the conjecture for $k = 3, s > s_0$.

Let us mention that the best general bound, true for all k, s and $n \ge k(s + 1)$ is due to the author (cf. [7] or [8]) and it says

$$|\mathcal{F}| \le s\binom{n-1}{k-1}. \tag{1.1}$$

Note that for $n = k(s + 1)$, (1.1) reduces to $|\mathcal{F}| \le |\mathcal{A}_k|$. This special case, the first non-trivial instance of the conjecture, was proved implicitly by Kleitman [14]. The case $s = 1$ of (1.1) is the classical Erdős–Ko–Rado Theorem [5].

2. Notation, tools

For a family $\mathcal{H} \subset 2^{[n]}$ and an element $i \in [n]$ we define $\mathcal{H}(i)$ and $\mathcal{H}(\bar{i})$ by

$$\begin{aligned} \mathcal{H}(i) &= \left\{H - \{i\} : i \in H \in \mathcal{H}\right\}, \\ \mathcal{H}(\bar{i}) &= \left\{H \in \mathcal{H} : i \notin H\right\}. \end{aligned}$$

For a subset $H = \{h_1, \dots, h_q\}$ we denote it also by $(h_1, \dots, h_q)$ whenever we know for certain that $h_1 < h_2 < \cdots < h_q$. For subsets $H = (h_1, \dots, h_q)$, $G = (g_1, \dots, g_q)$ we define the partial order, $\ll$ by

$$H \ll G \quad \text{iff} \quad h_i \le g_i \quad \text{for } 1 \le i \le q.$$

Definition 2.1. The family $\mathcal{F} \subset \binom{[n]}{k}$ is called stable if $G \ll F \in \mathcal{F}$ implies $G \in \mathcal{F}$.

In Frankl [7] (cf. also [8]) it was proved that it is sufficient to prove the Matching conjecture for stable families. Therefore throughout the paper we assume that $\mathcal{F}$ is stable and use stability without restraint.

An easy consequence of stability is the following. Let $\mathcal{F} \subset \binom{[n]}{k}$, $\nu(\mathcal{F}) = s$ and define $\mathcal{F}_0 = \left\{F \cap [ks + k - 1] : F \in \mathcal{F}\right\}$. Note that $\mathcal{F}_0$ is not k-uniform in general.

Proposition 2.1. $\nu(\mathcal{F}_0) = s$.

Proof. Suppose for contradiction that $G_1, \dots, G_{s+1} \in \mathcal{F}_0$ are pairwise disjoint and $F_1, \dots, F_{s+1} \in \mathcal{F}$ are such that $F_i \cap [ks + k - 1] = G_i$, $1 \le i \le s + 1$. Suppose further that $F_1, \dots, F_{s+1}$ are chosen subject to the above condition to minimize

$$\sum_{1 \le i < j \le s+1} |F_i \cap F_j|. \tag{2.1}$$

Since $\nu(\mathcal{F}) < s + 1$, the above minimum is positive. We establish the contradiction by showing that one can diminish it.

Choose some $x \in F_i \cap F_j$. Since $G_i \cap G_j = \emptyset$, $x \ge k(s + 1)$. Consequently, $|G_1| + \cdots + |G_{s+1}| \le k(s + 1) - 2 < ks + k - 1$. Thus we can choose $y \in [ks + k - 1]$ with $y \notin G_\ell$ for $1 \le \ell \le s + 1$. Now replace F_i by $F_i' = (F_i - \{x\}) \cup \{y\}$. Then $F_i' \ll F_i$, implying $F_i' \in \mathcal{F}$.

The intersections $F_j \cap [ks + k - 1]$, $j = 1, \dots, s + 1, j \ne i$ and $F_i' \cap [ks + k - 1]$ are still disjoint but the value of (2.1) is smaller. □

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