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On 3-uniform hypergraphs without a cycle of a given length

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ABSTRACT

We study the maximum number of hyperedges in a 3-uniform hypergraph on n vertices that does not contain a Berge cycle of a given length ℓ. In particular we prove that the upper bound for C_{2k+1}-free hypergraphs is of the order $O(k^2 n^{1+1/k})$, improving the upper bound of Győri and Lemons (2012) by a factor of $\Theta(k^2)$. Similar bounds are shown for linear hypergraphs.

1. A generalization of the Turán problem

Counting substructures is a central topic of extremal combinatorics. Given two (hyper)graphs G and H let $N(G;H)$ denote the number of subgraphs of G isomorphic to H. (Usually we consider a labeled host graph G.) Note that $N(G;K_2) = e(G)$, the number of edges of G. More generally, $N(\mathcal{G};H)$ is the maximum of $N(G;H)$ where $G \in \mathcal{G}$, a class of graphs. In most cases, in Turán type problems, $\mathcal{G}$ is a set of n-vertex $\mathcal{F}$-free graphs, where $\mathcal{F}$ is a collection of forbidden subgraphs. This maximum is denoted by $N(n, \mathcal{F};H)$. So $N(n, \mathcal{F};H)$ is the maximum number of copies of H in an $\mathcal{F}$-free graph on n vertices. The Turán number $\mathrm{ex}(n, \mathcal{F})$ is defined as $N(n, \mathcal{F};K_2)$. Let $\mathrm{ex}(m, n, \mathcal{F})$ be the maximum number edges in a bipartite graph with parts of order m and n vertices that do not contain any member of $\mathcal{F}$. $\mathcal{C}_\ell$ is the family of all cycles of length at most ℓ. For any graph G and any vertex x, we let $t(G)$ and $t(x)$ denote the number of triangles in G and the number of triangles containing x, respectively. Let $t_\ell(n) := N(n, \mathcal{C}_\ell;K_3)$.

Our starting point is the Bondy–Simonovits [3] theorem, $\mathrm{ex}(n, C_{2k}) \le 100kn^{1+1/k}$. Recall two contemporary versions due to Pikhurko [15], Bukh and Z. Jiang [4], respectively, and a classical result by Kővári, T. Sós, and Turán [14]. For all $k \ge 2$ and $n \ge 1$, we have

$$\mathrm{ex}(n, C_{2k}) \le (k-1)n^{1+1/k} + 16(k-1)n, \tag{1}$$

$$\mathrm{ex}(n, C_{2k}) \le 80\sqrt{k \log k}\, n^{1+1/k} + 10k^2 n, \tag{2}$$

$$\mathrm{ex}(n, n, C_4) \le n^{3/2} + 2n. \tag{3}$$

Erdős [6] conjectured that a triangle-free graph on n vertices can have at most $(n/5)^5$ five cycles and that equality holds for the blown-up C_5 if $5|n$. Győri [9] showed that a triangle-free graph on n vertices contains at most $c(n/5)^5$ copies of

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C_5, where $c < 1.03$. Grzesik [8], and independently, Hatami et al. [13] confirmed that Erdős' conjecture is true by using Razborov's method of flag algebras, i.e., $N(n, C_3; C_5) \le (n/5)^5$.

Bollobás and Győri [2] asked a related question: how many triangles can a graph have if it does not contain a C_5. They obtained the upper bound $t_5(n) \le (1+o(1))(5/4)n^{3/2}$ which yields the correct order of magnitude.

Later, Győri and Li [12] provided bounds on $t_{2k+1}(n)$.

$$\binom{k}{2} \mathrm{ex}\left(\frac{n}{k+1}, \frac{n}{k+1}, \mathcal{C}_{2k}\right) \le t_{2k+1}(n) \le \frac{(2k-1)(16k-2)}{3} \mathrm{ex}(n, C_{2k}). \tag{4}$$

The construction showing the lower bound in (4) is defined by considering a balanced bipartite (X, Y)-graph G on $2n/(k+1)$ vertices which is extremal not containing any members of $\mathcal{C}_{2k}$. Each vertex x in X is replaced by k vertices and connected to each other and to all neighbors of x, thus creating $\binom{k}{2}$ distinct triangles per each edge of G.

In Section 3 we improve the upper bound by a factor of $\Omega(k)$.

Theorem 1. *For $k \ge 2$,*

$$t_{2k+1}(n) := N(n, C_{2k+1}; K_3) \le 9(k-1)\, \mathrm{ex}\left(\left\lceil \frac{n}{3} \right\rceil, \left\lceil \frac{n}{3} \right\rceil, C_{2k}\right), \tag{5}$$

$$t_{2k}(n) \le \frac{2k-3}{3} \mathrm{ex}(n, C_{2k}). \tag{6}$$

The inequalities (1), (3) and (5) give $t_{2k+1}(n) \le 9(k-1)^2\, ((2/3)n)^{1+1/k} + O(n)$ for $k \ge 3$ and $t_5(n) \le \sqrt{3}n^{3/2} + O(n)$. This latter one is not better than the Bollobás–Győri bound. However, our constant factor in Theorem 1 is the best possible in the following sense. It is widely believed that the Turán numbers in the above statements are 'smooth', i.e., there are constants a_k, b_k depending only on k such that $\mathrm{ex}(n, n, C_{2k}) = (a_k + o(1))n^{1+1/k}$ and $\mathrm{ex}(n, n, \mathcal{C}_{2k}) = (b_k + o(1))n^{1+1/k}$. If these are indeed true then the ratio of the upper bound in (5) and the lower bound in (4) is bounded by a constant factor of $O(a_k/b_k)$. It is also believed that the sequence a_k/b_k is bounded (as $k \to \infty$), so further essential improvement is probably not possible.

Since the first version of this manuscript (2011) Alon and Shikhelman [1] improved the upper bound in Theorem 1 by a constant factor to $(16/3)(k-1)\,\mathrm{ex}(\lceil n/2 \rceil, C_{2k})$ and showed that $t_5(n) \le (1+o(1))(\sqrt{3}/2)n^{3/2}$. Nevertheless, we include our proof in Section 3 for completeness, and because we use Theorem 1 in our main result in the next section.

2. Berge cycles

A *Berge cycle* of length k is a family of distinct hyperedges $H_0, \ldots, H_{k-1}$ such that there are distinct vertices $v_0, \ldots, v_{k-1}$ satisfying

$$v_i v_{i+1} \subset H_i \text{ for } 0 \le i \le k-1 \pmod{k}.$$

A hypergraph is *linear*, also called nearly disjoint, if every two edges meet in at most one vertex. Let $C_\ell^{(3)}$ be the collection of 3-uniform Berge cycles of length ℓ.

We write $\mathrm{ex}_r(n, \mathcal{F})$ ($\mathrm{ex}_r^{\mathrm{lin}}(n, \mathcal{F})$, resp.) to denote the maximum number of hyperedges in a r-uniform (and linear, resp.) hypergraph on n vertices that does not contain any member of $\mathcal{F}$. Győri and Lemons [10] showed that

$$\mathrm{ex}\left(\left\lfloor \frac{n}{3} \right\rfloor, \left\lfloor \frac{n}{3} \right\rfloor, \mathcal{C}_{2k}\right) \le \mathrm{ex}_3(n, C_{2k+1}^{(3)}) < 4k^4 n^{1+\frac{1}{k}} + 15k^4 n + 10k^2 n. \tag{7}$$

The order of magnitude of the upper bound probably cannot be improved (as k is fixed and $n \to \infty$).

Győri and Lemons [11] extended their result to $C_{2k}^{(3)}$-free 3-uniform hypergraphs (and also to m-uniform hypergraphs) by showing that the same lower bound as in (7) holds for $\mathrm{ex}_3(n, C_{2k}^{(3)})$ and that $\mathrm{ex}_3(n, C_{2k}^{(3)}) \le c(k)n^{1+\frac{1}{k}}$. The construction showing the lower bound in (7) is defined by considering a balanced bipartite graph G on $n/3+n/3$ vertices which is extremal not containing any members of $\mathcal{C}_{2k}$. A 3-uniform $C_{2k}^{(3)}$-free hypergraph $\mathcal{H}$ is formed by doubling each vertex in one of the parts of G, thus turning each edge of G to a hyperedge of $\mathcal{H}$. The number of hyperedges in $\mathcal{H}$ is $e(G) = \mathrm{ex}(n/3, n/3, \mathcal{C}_{2k})$.

In this paper, we make improvements on the bounds on $\mathrm{ex}_3(n, C_{2k+1}^{(3)})$ and $\mathrm{ex}_3(n, C_{2k}^{(3)})$. First, observe that trivially

$$t_{2k+1}(n) \le \mathrm{ex}_3(n, C_{2k+1}^{(3)}). \tag{8}$$

(Consider the triple system defined by the triangles of a C_{2k+1}-free graph.) So (4) gives a lower bound which (probably) improves the lower bound in (7) by a factor of $\Omega(k)$.

The aim of this paper is to improve the upper bound in (7) by a factor of (at least) $\Omega(k^2)$ and also to simplify the original proof. In Section 4 we reduce the upper bound into three subproblems as follows.