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Strict chordal and strict split digraphs☆

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ABSTRACT

We introduce new versions of chordal and split digraphs, and explore their similarity with the corresponding undirected notions.

1. Introduction

A graph G is *chordal* if its vertices can be linearly ordered by $<$ so that for any $u < v < w$, if $u \sim v$ and $u \sim w$ then $v \sim w$. Such an ordering $<$ is called a *perfect elimination ordering of* G. Chordal graphs are an important class of perfect graphs, admit elegant recognition algorithms and characterizations, efficient optimization algorithms, and interesting applications [17]. They arise, for example, from consideration of sparse symmetric matrices whose sparseness can be preserved during Gaussian elimination [34]. A graph is chordal if and only if it does not have an induced cycle of length greater than three [17]. Chordal graphs can be recognized in linear time [34]; in fact the algorithm can be made certifying, in the sense that it either finds a simplicial ordering or an induced cycle of length greater than three [35]. The chromatic number, as well as the size of a maximum clique and maximum independent set in a chordal graph can be found in linear time, from the perfect elimination ordering [15,17].

Even nicer properties hold for its subclass of split graphs. A graph is *split* if its vertex set can be partitioned into a clique and an independent set [13]. A graph G is split if and only if both G and $\overline{G}$ are chordal [17]. It can be seen from this that a graph is split if and only if it does not have an induced C_4, C_5, or $\overline{C_4}$ [13]. Further, a graph is split if and only if its vertices can be linearly ordered by $<$ so that for any $u < v < w$, if $u \sim v$ then $v \sim w$ [10]. Whether or not an input graph is split can be recognized in linear time just by considering its degree sequence [17,13]. A certifying linear time recognition algorithm for split graphs is given in [24,23]. It is known that the proportion of chordal graphs on n vertices that are split graphs tends to one as n increases [2].

Perhaps the best known subclass is the class of interval graphs, since it has the most numerous natural applications [3,8]. A graph G is an *interval graph* if its vertices v can be represented by intervals I_v so that $v \sim w$ if and only if I_v intersects I_w [17]. A graph is an interval graph if and only if its vertices can be linearly ordered by $<$ so that for any $u < v < w$, if $u \sim w$

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then $v \sim w$ [17]. (It follows that an interval graph is necessarily chordal.) A forbidden induced subgraph characterization appears in [28] (see also [16,14]), and there are several linear-time recognition algorithms [5,25,19,9], and certifying algorithms [26].

All these graph classes are contained in the class of perfect graphs. A graph G is *perfect* if the chromatic number and the maximum clique size are equal for G and all of its induced subgraphs. The class of perfect graphs is considered of prime importance in algorithmic graph theory [4,17,7], a prototype framework for combinatorial max–min results. A graph is perfect if and only if it does not contain an induced cycle of odd length greater than three, or its complement [4,7]. Perfect graphs can be recognized in polynomial time [6], and the basic optimization problems of finding the chromatic number, the clique number, and the independence number can be solved in polynomial time [18].

There have been attempts to translate the elegance of these results to the realm of digraphs.

Perhaps the best known is the digraph analogue of interval graphs, defined as follows [11]. (A different definition was proposed in [20]; it only yields acyclic digraphs.) A digraph D is an *interval digraph* if its vertices v can be represented by pairs of intervals I_v, J_v, so that $v \to w$ if and only if I_v intersects J_w. Some characterizations of interval digraphs are known [11], especially in terms of their adjacency matrices; however there is no forbidden induced subgraph (or substructure) characterization known, and the existing polynomial time recognition algorithms have high-degree-polynomial time bounds [31]. (We note that a low-degree-polynomial time algorithm is claimed in [33].) Interestingly, there is a more convenient subclass of interval digraphs. In [12] the authors define a digraph D to be an *adjusted interval digraph* if its vertices v can be represented by pairs of intervals I_v, J_v, so that $v \to w$ if and only if I_v intersects J_w, where the intervals I_v and J_v have the same left endpoint (are "left-adjusted"). It turns out that adjusted interval digraphs show a greater similarity to interval graphs than plain interval digraphs. In particular, they have a forbidden structure characterization, an ordering characterization, and a low-degree-polynomial time recognition algorithm [12]. It turned out that interval digraphs were too ambitious a generalization of interval graphs, and by being more modest (while still making a large digraph generalization of interval graphs), we obtain a better analogue.

Chordal digraphs are defined in [30,22], also in a way to correspond to sparse general matrices (not necessarily symmetric as above), whose sparseness can be preserved during Gaussian elimination. (Again a different definition tailored to acyclic digraphs was proposed in [21].) Specifically, a digraph D is a *chordal digraph* if its vertices can be linearly ordered by $<$ so that for any $u < v < w$, if $v \to u$ and $u \to w$ then $v \to w$ [30,22]. Some natural analogues of the results about chordal graphs are obtained for the class of these chordal digraphs [30]. However, a forbidden subgraph characterization is not known, except in certain very special cases [30,21,29].

Split digraphs were defined in [27] as follows. D is a *split digraph* if its vertices can be partitioned into four sets A, B, C, D where A is a strong clique (a complete symmetric digraph), D is an independent set, all possible arcs go from A to C and from B to A and C, and no arcs go from C to B or D and from D to B. Moreover, it is required that the partition does not place all vertices in B or all vertices in C. (Otherwise every digraph would be split.) The main appeal of these split digraphs seems to be that can be recognized by just considering their degree sequence, just as split graphs can [27]. However, this class of split digraphs is not closed under taking induced subgraphs; any digraph D can be made split by the addition of one new vertex dominating all vertices of D.

It appears that these digraph analogues of chordal and split graphs may again be too general to recover the elegance of chordal and split graphs. In this paper, we propose two new notions of chordal and split digraphs, which we call *strict chordal digraphs* and *strict split digraphs*. These properly generalize the undirected cases, in the sense that a chordal (respectively split) graph, when viewed as a digraph, by replacing each edge uv by the two arcs uv and vu, is a strict chordal (respectively strict split) digraph. There is however a much greater number of strict chordal and strict split digraphs that do not arise this way from their undirected analogues (see the Concluding Remarks). Nevertheless, the underlying graph of any strict chordal digraph is a chordal graph, and the underlying graph of any strict split digraph is a split graph. Moreover our classes turn out to be close enough to the undirected notions to allow extending some of the fundamental theorems from the undirected case, including forbidden subgraph characterizations and polynomial time recognition algorithms (linear time in the case of strict split graphs).

The class of perfect graphs also has a digraph analogue. For digraphs, D is a *perfect digraph* [1] if for D and every induced subgraph of D, the dichromatic number and the clique number are equal. (The dichromatic number of D is the smallest number of colours for which D can be coloured so that each colour induces an acyclic subgraph; and the clique number of D is the number of vertices in a largest strong clique.) The authors of [1] have succeeded in obtaining many results analogous to the undirected cases, including a forbidden subgraph characterization, and polynomial time optimization algorithms for some basic problems. However, the recognition of perfect digraphs is $\mathcal{NP}$-complete [1].

It will turn out that strict chordal digraphs and (therefore) the strict split digraphs are perfect in this sense.

As used above, in a graph G we write $u \sim v$ to mean there is an edge joining u and v in G; we also say that u and v are *adjacent*. Similarly, in a digraph D, $u \to v$ means there is an arc from u to v; we also say that u *dominates* v. In addition to these standard terms we also use the following helpful terminology, in a digraph D. We say that u and v are *adjacent* in (a digraph) D if $u \to v$ or $v \to u$ (or both); in analogy with graphs we write $u \sim v$ in this case. We say that u and v are *strongly adjacent* in D if $u \to v$ and $v \to u$; in this case we write $u \leftrightarrow v$. (We also call $u \leftrightarrow v$ a *digon* or a *symmetric arc*.) We say that u and v are *weakly adjacent* in D if either $u \to v$ or $v \to u$; in this case we write $u \div v$. (In the special case when $u \to v$ but not $v \to u$ we write $u \mapsto v$ and similarly for $v \mapsto u$.)