Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Minimum multiple originator broadcast graphs☆

Arthur L. Liestman[a], Dana Richards[b,*]

[a] School of Computing Science, Simon Fraser University, Burnaby, BC V5A 1S6, Canada

[b] Department of Computer Science, George Mason University MSN 4A5, 4400 University Drive, Fairfax, VA 22030, USA

ARTICLE INFO

Article history:
Received 2 March 2015
Received in revised form 9 March 2016
Accepted 20 June 2016
Available online xxxx

Dedicated to Levon Khachatrian

Keywords:
Broadcast
Multiple originator
Minimum broadcast graph

ABSTRACT

Broadcasting from multiple originators is a variant of broadcasting in which any k vertices may be the originators of a message in a network of n vertices. A minimum broadcast graph has the fewest possible edges while still allowing minimum time broadcasting from any set of k originators. We provide a census of all known such graphs.

1. Introduction

Broadcasting is the process of message dissemination in a communication network in which a message, originated by one vertex, is transmitted to all vertices of the network by placing a series of calls over the communication lines of the network. This is to be completed as quickly as possible. Typically, it is assumed that each call involves only one informed vertex and one of its neighbors, each call requires one unit of time, a vertex can participate in only one call per unit of time, and a vertex can only call its neighbors. Here, we consider broadcasting from any set of k originators.

Given a connected graph $G = (V, E)$ and a subset of the vertices $V' \subseteq V$ the *broadcast time of the set* V', $b(V')$, is the minimum number of time units required to complete a broadcast from the vertices V'. Since the number of informed vertices can at most be doubled during each time unit, it is clear that for any set V' of k vertices in a connected graph G with n vertices, $b(V') \geq t(n, k)$, where $t(n, k) = \lceil \log_2 \frac{n}{k} \rceil$. The *k-originator broadcast time of a graph G*, denoted $b_k(G)$, is the maximum broadcast time of any such subset V' in G, with $|V'| = k$, i.e. $b_k(G) = \max\{b(V') | V' \subseteq V, |V'| = k\}$. We use the term *k-originator broadcast graph* to refer to any graph G on n vertices with $b_k(G) = t(n, k)$. The *k-originator broadcast function*, $B_k(n)$, is the minimum number of edges in any k-originator broadcast graph on n vertices. A *minimum k-originator broadcast graph* is a k-originator broadcast graph on n vertices having $B_k(n)$ edges.

Early work by Khachatrian appeared in [10,11]. For surveys of results on broadcasting and related problems, see Hedetniemi, Hedetniemi and Liestman [8], Fraigniaud and Lazard [5], Hromkovič, Klasing, Monien, and Peine [9] and Harutyunyan, Liestman, Peters and Richards [7]. More recent results are [2,1,6].

In this paper we will provide a census of multiple originator minimum broadcast graphs, for various values of k and n where they have been determined. The upper bounds are given by exhibiting such a graph and explaining how it works for

☆ This work was supported by the Natural Sciences and Engineering Research Council of Canada, grant OGP0001734.

* Corresponding author.
E-mail address: richards@cs.gmu.edu (D. Richards).

http://dx.doi.org/10.1016/j.dam.2016.06.015
0166-218X/

any subset of k vertices. The lower bounds can involve a detailed case analysis. To aid the reader these are grouped by the time necessary to broadcast. Table 1, a summary of our theorems, appears at the end of the paper.

2. Background

The concept of multiple originators has appeared in various restricted cases, such as the best way to position originators in a tree [4]. We however focus on the worst-case problem of unrestricted placement of originators. We have previously given asymptotic results on $B_k(n)$ [13]. In particular we have these upper and lower bounds, where $L(x)$ be the number of leading 1 bits in the binary representation of the integer x.

Theorem 2.1. *For every* $n \geq 1$, B_k *is* $\Omega\left(L(\lceil \frac{n}{k} \rceil - 1) \cdot (n-k)\right)$.

Theorem 2.2. *For every* $n > 1$, $B_k(n)$ *is* $O\left(L(\lceil \frac{n}{k} \rceil - 1) \cdot n \cdot k\right)$.

Since these bounds are not tight we directed our efforts in a very different direction. (They are tight for constant k but general interest is for variable numbers.) In this paper we consider the cases where we can prove $B_k(n)$ exactly. This will involve much detail and constrain us to small values of k and n.

A census of minimum broadcast graphs has previously been undertaken for $k = 1$. Farley, Hedetniemi, Mitchell and Proskurowski [3,14] presented such graphs for all $n \leq 15$. These have been extended over a series of publications as discussed in the survey papers. Some papers address a single value of n [12,15]. At this point $B_1(n)$ is known for these values of n: 1 to 22, 26 to 32, 58 to 64, and similar ranges below powers of two [7].

We will organize our search by considering examples which allow only two time units to broadcast, and then we will consider examples for three time units. When the broadcast must be completed in one time unit (when $k \geq \frac{n}{2}$) we have an exact value for $B_k(n)$ [13].

Theorem 2.3. $B_k(n) = \lceil \frac{n(n-k)}{2} \rceil$ *for* $k \geq \frac{n}{2}$.

We state some general results that will be used below.

Lemma 2.4. *If* $t(n, k) = t(n, k+1)$, *then* $B_k(n) \geq B_{k+1}(n)$.

Proof. This follows from the observation that if any set of k originators can broadcast in graph G in time t, then any set of $k+1$ originators can also broadcast in G in time t. □

Lemma 2.5. *If* $t(n, k) = t(n+1, k)$ *and a minimum* k *originator broadcast graph on* $n+1$ *vertices has a vertex of degree* 2, *then* $B_k(n) \leq B_k(n+1) - 1$.

Proof. Let G be such a graph, let a be such a vertex of degree 2 and let b and c be its neighbors. Let G' be the graph on n vertices formed by deleting a and its incident edges and adding the edge between b and c. G' has fewer edges than G. In any broadcasting scheme for k originators (not including a), if one neighbor, say b, calls a and a later calls c, replace the former call with a call from b to c. Otherwise, delete any calls involving a. The result is a valid k originator broadcasting scheme for G'. □

Similar results hold for higher degree vertices, such as:

Lemma 2.6. *If* $t(n, k) = t(n+1, k)$ *and a minimum* k *originator broadcast graph on* $n+1$ *vertices has a vertex of degree* 3, *then* $B_k(n) \leq B_k(n+1)$.

Proof. Let G be such a graph, let a be such a vertex of degree 3 and let b, c and d be its neighbors. Let G' be the graph on n vertices formed by deleting a and its incident edges and adding edges between b, c and d. G' has at most the same number of edges as G. In any broadcasting scheme for k originators (not including a), if b calls a and a later calls c and then (perhaps) d, replace the call from b to a with a call from b to c and replace the call from a to c with a call from c to d. Otherwise, delete any calls involving a. The result is a valid k originator broadcasting scheme for G'. □

3. Specific values for time 2

In this section we only discuss results for cases where $t(n, k) = 2$. The reader will not be reminded repeatedly that only two time units are allowed. This section is a case study for determining exact values of $B_k(n)$ for small values of n. Note that in two time units, each originator can inform at most three other vertices and it must have at least two non-originator neighbors to inform three. The theorems in this section are presented in an order that corresponds to filling in columns of Table 1 (at the end), which naturally leads to building upon previous proofs.

To prove lower bounds we need two lemmas. A vertex u is *isolated* if there are k vertices more than two steps away from u. This lemma follows since such a vertex cannot be informed if the originators are all too far away.

Please cite this article in press as: A.L. Liestman, D. Richards, Minimum multiple originator broadcast graphs, Discrete Applied Mathematics (2016), http://dx.doi.org/10.1016/j.dam.2016.06.015