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Free monoids and forests of rational numbers

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ABSTRACT

The Calkin–Wilf tree is an infinite binary tree whose vertices are the positive rational numbers. Each such number occurs in the tree exactly once and in the form a/b, where a and b are relatively prime positive integers. This tree is associated with the matrices $L_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $R_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, which freely generate the monoid $SL_2(\mathbf{N}_0)$ of 2×2 matrices with determinant 1 and nonnegative integral coordinates. For other pairs of matrices L_u and R_v that freely generate submonoids of $GL_2(\mathbf{N}_0)$, there are forests of infinitely many rooted infinite binary trees that partition the set of positive rational numbers, and possess a remarkable symmetry property.

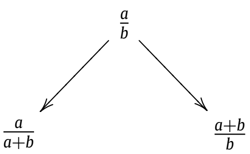
1. The Calkin–Wilf tree of rational numbers

A directed graph is a *rooted infinite binary tree* if it is a tree with the following properties:

(i) Every vertex is the tail of exactly two edges. Equivalently, every vertex has *outdegree* 2.
(ii) There is a vertex z such that every vertex $v \neq z$ is the head of exactly one edge, but z is not the head of any edge. Equivalently, every vertex $v \neq z$ has *indegree* 1, and z has indegree 0. We call z the *root* of the tree.
(iii) The graph is connected.

In this paper, a *forest* is a directed graph whose connected components are rooted infinite binary trees.

Let $\mathbf{Q}^+$ denote the set of positive rational numbers. We call the rational number a/b *reduced* if $b \geq 1$ and the integers a and b are relatively prime. The Calkin–Wilf tree [6] is a rooted infinite binary tree whose vertex set is the set of positive reduced rational numbers, and whose root is 1. In this tree, every positive reduced rational number a/b is the tail of two edges. The heads of these edges are the positive rational numbers $a/(a+b)$ and $(a+b)/b$. We draw this as follows:



with $a/(a+b)$ on the left and $(a+b)/b$ on the right. Note that

$$0 < \frac{a}{a+b} < 1 < \frac{a+b}{b}.$$

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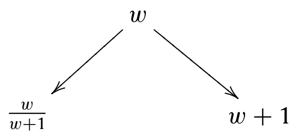
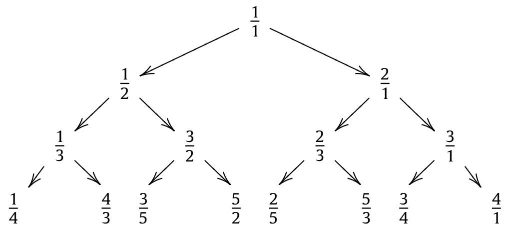
Equivalently, if $w = a/b$, then the generation rule of the tree is



(1)

Calkin and Wilf [6] introduced this enumeration of the positive rationals in 2000. It is related to the Stern–Brocot sequence [5,21], discussed in [10], and has stimulated much recent research (e.g. [1,3,4,7,9,14–16,19]). For work related to this paper, see [11,12].

The first four rows the Calkin–Wilf tree are as follows:



We enumerate the numbers on the rows of the Calkin–Wilf tree as follows. Row 0 contains only the number 1. Row 1 contains the numbers 1/2 and 2. For every nonnegative integer n, the nth row of the Calkin–Wilf tree contains 2^n positive reduced rational numbers. The nth row of the tree is also called the *nth generation* of the tree. We denote the ordered sequence of elements of the nth row, from left to right, by $c(n, 1), c(n, 2), \ldots, c(n, 2^n)$. For example, $c(2, 3) = 2/3$ and $c(3, 6) = 5/3$. Note that $0 < c(n, 2i - 1) < 1 < c(n, 2i)$ for $i = 1, 2, \ldots, 2^{n-1}$.

Here are four properties of the Calkin–Wilf tree:

(i) Symmetry formula: For every nonnegative integer n and for $i = 1, \ldots, 2^n$,

$$c(n, i)c(n, 2^n + 1 - i) = 1.$$

The proof is by induction on n.

(ii) Denominator–numerator formula: For every positive integer n, we have $c(n, 1) = 1/(n + 1)$ and $c(n, 2^n) = n + 1$. For $j = 1, \ldots, 2^n - 1$, if $c(n, j) = p/q$, then $c(n, j + 1) = q/r$. Thus, as we move through the Calkin–Wilf tree from row to row, and from left to right across each row, the denominator of each fraction in the tree is the numerator of the next fraction in the tree. This is in Calkin–Wilf [6].

(iii) Successor formula: For every positive integer n and for $j = 1, \ldots, 2^n - 1$, we have

$$c(n, j + 1) = \frac{1}{2[c(n, j)] + 1 - c(n, j)}$$

where $[x]$ denotes the integer part of the real number x. This result is due to Moshe Newman [2,18].

(iv) Row formula: Let a/b be a positive reduced rational number. If

$$\frac{a}{b} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_{k-1} + \frac{1}{a_k}}}}$$
$$= [a_0, a_1, \ldots, a_{k-1}, a_k]$$

is the finite continued fraction of a/b, then a/b appears on the nth row of the Calkin–Wilf tree, where $n = a_0 + a_1 + \cdots + a_{k-1} + a_k - 1$. This is discussed in Gibbons, Lester, and Bird [8].

2. Freely generated monoids and a symmetry of trees

A monoid is a semigroup with an identity. Let $GL_2(\mathbf{R}_{\geq 0})$ denote the multiplicative monoid of 2×2 matrices with nonzero determinant and with coordinates in the set $\mathbf{R}_{\geq 0}$ of nonnegative real numbers. To every matrix

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \in GL_2(\mathbf{R}_{\geq 0})$$

we associate the linear fractional transformation

$$A(w) = \frac{a_{1,1}w + a_{1,2}}{a_{2,1}w + a_{2,2}}.$$