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On the Shortest Path Game[☆]

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ABSTRACT

In this work we address a game theoretic variant of the shortest path problem, in which two decision makers (players) move together along the edges of a graph from a given starting vertex to a given destination. The two players take turns in deciding in each vertex which edge to traverse next. The decider in each vertex also has to pay the cost of the chosen edge. We want to determine the path where each player minimizes its costs taking into account that also the other player acts in a selfish and rational way. Such a solution is a subgame perfect equilibrium and can be determined by backward induction in the game tree of the associated finite game in extensive form.

We show that the decision problem associated with such a path is PSPACE-complete even for bipartite graphs both for the directed and the undirected version. The latter result is a surprising deviation from the complexity status of the closely related game Geography.

On the other hand, we can give polynomial time algorithms for directed acyclic graphs and for cactus graphs even in the undirected case. The latter is based on a decomposition of the graph into components and their resolution by a number of fairly involved dynamic programming arrays. Finally, we give some arguments about closing the gap of the complexity status for graphs of bounded treewidth.

1. Introduction

We consider the following game on a graph: There is a directed graph $G = (V, A)$ given with vertex set V and arc set A with positive costs $c(u, v)$ for each arc $(u, v) \in A$ and two designated vertices $s, t \in V$. The aim of Shortest Path Game is to find a directed path from s to t in the following setting: The game is played by two players A and B who have full knowledge of the graph. They start in s and always move together along arcs of the graph. In each vertex the players take turns to select the next vertex to be visited among all neighboring vertices of the current vertex with player A taking the first decision in s. The player deciding in the current vertex also has to pay the cost of the chosen arc. Each player wants to minimize the total arc costs it has to pay. The game continues until the players reach the destination vertex t.[1] Later, we will also consider the same problem on an undirected graph $G = (V, E)$ with edge set E which is quite different in several aspects.

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[1] We will always assume that a path from s to t exists.

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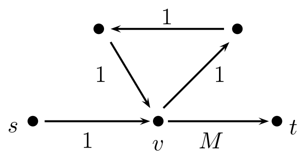
We will impose two restrictions on the setting in order to ensure that vertex t is in fact reachable and to guarantee finiteness of the game. Even in a connected graph, it is easy to see that the players may get stuck at some point and reach a vertex where the current player has no emanating arc to select. To avoid such a deadlock, we restrict the players in every decision to choose an arc (or edge, in the undirected case) which still permits a feasible path from the current vertex to the destination t (which is computationally easy to check).

(R1) No player can select an arc which does not permit a path to vertex t.

In the undirected case, it could be argued that for a connected graph this restriction is not needed since the players can always leave dead ends again by going back the path they came. Clearly, this would imply cycles of even length. Such cycles, however, may cause the game to be infinite as illustrated by the following example.

Example 1. Consider the graph depicted below. Player B has to decide in vertex v whether to pay the cost $M \gg 2$ or enter the cycle of length 3. In the latter case, the players move along the cycle and then A has to decide in v with the same two options as before for player B. In order to avoid paying M both players may choose to enter the cycle whenever it is their turn to decide in v leading to an infinite game.



The general setting of the game is representable as a game in extensive form. However, there is no concept in game theory to construct an equilibrium for an infinite game in extensive form (only bargaining models such as the Rubinstein bargaining game are well studied, cf. Osborne [14], ch. 16). Thus, we have to impose reasonable conditions to guarantee finiteness of the game.

A straightforward idea would be to restrict the game to *simple paths* and request that each vertex may be visited at most once. While this restriction would lead to a simpler setting of the game, it would also rule out very reasonable strategic behavior, such as going through a cycle in the graph. Indeed, a cycle of even length cannot be seen as a reasonable choice for any player since it necessarily increases the total costs for both players. However, in Example 1 it would be perfectly reasonable for B to enter the cycle of *odd* length and thus switch the role of the decider in v. Therefore, a second visit of a vertex may well make sense. However, if also A enters the cycle in the next visit of v, two rounds through the odd cycle constitute a cycle of even length which we rejected before. Based on these arguments we will impose the following restriction which permits a rather general setting of the game:

(R2) The players cannot select an arc which implies necessarily a cycle of even length.

Note that (R2) implies that an odd cycle may be part of the solution path, but it may not be traversed twice (thus reversing the switching between the players) since this would constitute a cycle of even length. This aspect of allowing odd cycles will be taken explicitly into account in the presented algorithms. In the remainder of the paper we use "cycle" for any closed walk, also if vertices are visited multiple times. It also follows that each player can decide in each vertex at most once and any arc can be used at most once by each player.

A different approach to guarantee finiteness of the game would be to impose an upper bound on the total cost accrued by each player. This may seem relevant especially for answering the decision version where we ask whether the costs for both players remain below given bounds (see Section 2). Clearly, such an upper bound permits a clear answer to problems such as Example 1. In that case, the precise value of the bound defines which of the two players has to pay M after a possibly large number of rounds through the cycle. Anticipating this outcome, the unfortunate player will accept the cost M the first time it has to decide in v. However, it should be pointed out that a slight change in the upper bound may completely turn around the situation and cause the other player to pay M. Thus, the outcome of the game would depend on the remainder of the division of the bound by the cycle cost. This would cause a highly erratic solution structure and does not permit a consistent answer to the decision problem.

As an illustrative example of the game consider the following scenario. Two scientists A and B meet at a conference and lay out the plan for a joint paper. This requires the completion of a number of tasks. Each task requires a certain working time (identical for both A and B) to be completed. Moreover, there is a precedence structure on the set of tasks, e.g. the notation and definitions have to be finished before the formal statement of results, which in turn should precede the writing of the proofs, etc. Obviously, this implies a partial ordering on the set of tasks. We can represent any state of the paper by a vertex of a directed graph with an arc representing a feasible task whose completion leads to a new state (= vertex).

To avoid confusion and a mix-up of file versions the two scientists decide to alternate in working on the paper such that each of them finishes a task and then passes the file to the other co-author. Since none of them wants to patronize the other they have no fixed plan on the division of tasks, but each can pick any outstanding task as long as it is a feasible step to a new state of the paper. They continue in this way until the paper is finished, i.e. the final state is reached. Since both scientists are extremely busy each of them tries to minimize the time devoted to the paper taking into account the anticipated rational optimal decisions by the other.

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