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Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

New semidefinite programming relaxations for the Linear Ordering and the Traveling Salesman Problem

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ARTICLE INFO

Article history:
Received 5 December 2014
Received in revised form 8 July 2016
Accepted 12 July 2016
Available online xxxx

Keywords:
Linear Ordering Problem
Max cut problem
Traveling Salesman Problem
Target visitation problem
Vertex ordering problems
Semidefinite programming
Approximation algorithms
Global optimization

ABSTRACT

In 2004 Newman [43] suggested a semidefinite programming relaxation for the Linear Ordering Problem (LOP) that is related to the semidefinite program used in the Goemans–Williamson algorithm to approximate the Max Cut problem (Goemans and Williamson, 1995). Her model is based on the observation that linear orderings can be fully described by a series of cuts. Newman (2004) [43] shows that her relaxation seems better suited for designing polynomial-time approximation algorithms for the (LOP) than the widely-studied standard polyhedral linear relaxations.

In this paper we strengthen the relaxation proposed by Newman (2004) [43] and conduct a polyhedral study of the corresponding polytope. Furthermore we relate the relaxation to other linear and semidefinite relaxations for the (LOP) and for the Traveling Salesman Problem and elaborate on its connection to the Max Cut problem.

1. Introduction

For the past decades, combinatorial methods and linear programming (LP) techniques have partly failed to yield improved approximation guarantees for many well-studied vertex ordering problems such as the Linear Ordering Problem (LOP) or the Traveling Salesman Problem (TSP). Semidefinite programming (SDP) has proved to be a powerful tool for obtaining strong approximation results for a variety of cut problems, starting with the Max Cut problem [23].

SDP is the extension of LP to linear optimization over the cone of symmetric positive semidefinite matrices. This includes LP problems as a special case, namely when all the matrices involved are diagonal. A (primal) SDP can be expressed as the following optimization problem

$$\begin{aligned} &\inf_{X}\{\langle C, X\rangle : X \in \mathcal{P}\},\\ &\mathcal{P} := \{\, X \mid \langle A_i, X\rangle = b_i,\ i \in \{1,\ldots,m\},\ X \succcurlyeq 0\,\}, \end{aligned} \qquad \text{(SDP)}$$

where the data matrices A_i, $i \in \{1,\ldots,m\}$ and C are symmetric. For further information on SDP we refer to the handbooks [53,2] for a thorough coverage of the theory, algorithms and software in this area, as well as a discussion of many application areas where semidefinite programming has had a major impact.

SDP has been successfully applied to many other problems besides the Max Cut problem that can be considered as cut problems such as the dicut problem [16], coloring k-colorable graphs [36], maximum k-cut [19], maximum bisection and maximum uncut [27], to name a few. In contrast, there is no such comparably general approach for approximating vertex ordering problems.

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http://dx.doi.org/10.1016/j.dam.2016.07.013

In 2004 Newman [43] suggested an SDP relaxation for the (LOP) that is related to the semidefinite program used in the Goemans–Williamson algorithm to approximate the Max Cut problem [23]. She observes that linear orderings can be fully described by a series of cuts. Accordingly she suggests an SDP relaxation using cut variables[1] to approximate the (LOP). Newman [43] shows that her relaxation seems well-suited for designing polynomial-time approximation algorithms for the (LOP). In detail she proves that for sufficiently large n her SDP relaxation has an integrality gap of no more than 1.64 for a class of random graphs on n vertices. These random graphs include the graphs used in [45] to demonstrate integrality gaps of 2 for widely-studied polyhedral linear relaxations,

Due to the interesting connection between orderings and cuts and the promising theoretical results, we decided to study the SDP relaxation suggested by Newman [43] in more detail. In this paper we propose a formulation of the (LOP) using cut variables and strengthen the SDP relaxation of Newman [43] by studying the corresponding polytope. Furthermore we extend the relaxation to other vertex ordering problems and elaborate on its connection to the Max Cut problem. As our relaxations provide new polynomial-time convex approximations of the (LOP) and (TSP) with a rich mathematical structure, we hope that they may be helpful to improve approximation results for vertex ordering problems.

We refer to our companion paper [32] for an application of the theoretical results in this paper, i.e. we use the new semidefinite relaxations suggested here to design an exact SDP approach for the Target Visitation Problem (TVP) [25,31] that is as a combination of the (LOP) and the (TSP).[2]

The paper is structured as follows. In Section 2 we recall some basic facts about the (LOP) and consider different ways of modeling it. Section 3 is mainly devoted to the polyhedral study of the (LOP) model using cut variables. In Section 4 we show the exact relation of our model using cut variables to the Max Cut problem and show how to extend it to other vertex ordering problems. In Section 5 we use small but hard (LOP) and (TSP) instances to compare the strength of our relaxations proposed with other linear and semidefinite relaxations from the literature. Section 6 concludes the paper.

2. Linear and quadratic models for the Linear Ordering Problem

In this section first we briefly review the basic properties, state-of-the-art exact and heuristic approaches and main areas of application of the (LOP). Then we discuss linear and quadratic formulations of the (LOP) using ordering variables in Section 2.2. Finally in Section 2.3 we consider a quadratically constrained quadratic program using cut variables proposed by Newman [43] that gives an upper bound for the (LOP). We show how to adapt this program to obtain a quadratic formulation in cut variables for the (LOP) that forms the basis for the polyhedral study in Section 3.2.

2.1. A brief review on the Linear Ordering Problem

Ordering problems associate to each ordering (or permutation) of the set $[n] := \{1, 2, \ldots, n\}$ a profit and the goal is to find an ordering of maximum profit. In the simplest case of the (LOP), this profit is determined by those pairs $(u, v) \in [n] \times [n]$, where u comes before v in the ordering. Thus in its matrix version the (LOP) can be defined as follows. Given an $n \times n$ matrix $W = (w_{ij})$ of integers, find a simultaneous permutation π of the rows and columns of W such that

$$\sum_{\substack{i,j \in [n] \\ i<j}} w_{\pi(i),\pi(j)},$$

is maximized. Equivalently, we can interpret w_{ij} as weights of a complete directed graph G with vertex set $V = [n]$. A tournament consists of a subset of the arcs of G containing for every pair of vertices i and j either arc (i, j) or arc (j, i), but not both. Then the (LOP) consists of finding an acyclic tournament, i.e. a tournament without directed cycles, of G of maximum total edge weight. Let us further clarify this definition with the help of a toy example. We consider 4 vertices and the pairwise weights $w_{12} = w_{41} = w_{34} = 1$, $w_{31} = w_{24} = 2$. Fig. 1 illustrates the optimal ordering of the vertices and the corresponding benefit.

The (LOP) is well known to be NP-hard [20] and it is even NP-hard to approximate the (LOP) within the factor $\frac{65}{66}$ [45]. If all entries of W are nonnegative, a $\frac{1}{2}$-approximation is trivial: In any ordering of the vertices, either the set of forward edges or the set of backward edges accounts for at least half of the total edge weight. If a better approximation factor than half could be obtained using a polynomial-time algorithm, this would disprove the famous Unique Games Conjecture by Khot [38]. Furthermore Newman and Vempala [45] showed that widely-studied polyhedral linear relaxations for the (LOP) cannot be used to narrow the quite large gap $\left[\frac{1}{2}, \frac{65}{66}\right]$ (for more details see Section 2.2).

The (LOP) arises in a large number of applications in such diverse fields as economics (ranking and voting problems [50] and input–output analysis [40]), sociology (determination of ancestry relationships [21]), graph drawing (one sided crossing

[1] For a formal definition of the term "cut variables" we refer to the beginning of Section 2.3.

[2] We note that the content of the companion paper is as disjoint from this paper as possible: it designs an exact SDP approach to the (TVP) and its associated (military) applications. In the companion paper we omitted all proofs concerning the polyhedral properties and referred to this paper for details.