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Differential approximation schemes for half-product related functions and their scheduling applications

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ABSTRACT

We address the problem of minimizing a half-product function, with and without a linear knapsack constraint. We show how to convert known fully polynomial-time approximation schemes to differential approximation schemes that handle the problems with and without an additive constant and with and without a linear knapsack constraint. Thereby, we resolve the issue of differential approximation for a range of scheduling problems.

1. Introduction

The half-product function is a special form of a (pseudo) Boolean quadratic function that has been studied since the 1990s. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a vector with n Boolean components. Consider the function

$$h(\mathbf{x}) = \sum_{1 \le i < j \le n}^{n} \alpha_i \beta_j x_i x_j - \sum_{j=1}^{n} \gamma_j x_j, \tag{1}$$

where for each j, $1 \le j \le n$, the coefficients α_j, β_j and γ_j are non-negative integers. This function and the term "half-product" were introduced by Badics and Boros [3], who considered the problem of its minimization. The function $h(\mathbf{x})$ is called a *half-product* since its quadratic part consists of roughly half of the terms of the product $\left(\sum_{j=1}^{n} \alpha_j x_j\right)\left(\sum_{j=1}^{n} \beta_j x_j\right)$.

In this paper, we consider the problem of minimizing $h(\mathbf{x})$ as well as several related problems. Notice that we only are interested in the instances of the problem for which the minimum value of the function is strictly negative; otherwise, setting all decision variables to zero solves the corresponding problem.

Partly, the interest in the problems of minimizing the functions related to the half-product is due to their applications to scheduling problems with min-sum objective functions. Notice that in those applications a scheduling objective function usually is written in the form

$$f(\mathbf{x}) = h(\mathbf{x}) + K, \tag{2}$$

where K is a given additive constant; see [6,17,18] for reviews.

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Another range of scheduling applications is related to the problems of minimizing functions similar to $h(\mathbf{x})$ or $f(\mathbf{x})$ subject to a linear knapsack constraint $\sum_{j=1}^{n} \alpha_j x_j \le A$; see [17,18] for reviews. In the knapsack constraint, the value α_j can be understood as the weight of item j, $1 \le j \le n$, i.e., $x_j = 1$ means that item j is placed into a knapsack of capacity A, while $x_j = 0$ means that the corresponding item is not placed into the knapsack. Notice that the coefficients α_j in the knapsack constraint are the same as in the quadratic terms of the objective function.

For the introduced problems of Boolean programming, it is often convenient to reformulate the objective function not as a function of $0-1$ variables, but as a set-function.

We illustrate this for the half-product function $h(\mathbf{x})$ of the form (1). For a set $N = \{1, 2, \ldots, n\}$, let 2^N denote the family of all subsets of N. For an n-dimensional vector $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ define $p(S) = \sum_{j\in S} p_j$ for every non-empty set $S \in 2^N$ and define $p(\emptyset) = 0$.

Given a function $\varphi(\mathbf{x})$ with Boolean arguments $x_j \in \{0, 1\}$, we can associate it with a set-function $\varphi(S)$. More precisely, a Boolean vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ can be associated with a set $S \in 2^N$ in such a way that element $j \in N$ belongs to S if and only if $x_j = 1$. From now on, we see the Boolean and the set representation of a function as equivalent, and use both types of notation, $\varphi(\mathbf{x})$ and $\varphi(S)$, whichever is more convenient. See [7] for a detailed discussion of the link between set-functions and Boolean functions.

In the case of the half-product function, for a vector $\mathbf{x}$, we can rewrite $h(\mathbf{x})$ in the set-function form as

$$h(S) = \sum_{i,j\in S;\ i<j;} \alpha_i \beta_j - \gamma(S). \tag{3}$$

Then, the problem of minimizing $h(\mathbf{x})$ can be understood as the problem of finding a set-minimizer S_* such that the inequality $h(S_*) \le h(S)$ holds for all sets $S \in 2^N$. In a similar way, introduce a set-maximizer S^* such that $h(S^*) \ge h(S)$ holds for all sets $S \in 2^N$.

A set-function $\varphi(S)$ is called monotone non-decreasing if $\varphi(A) \le \varphi(B)$ holds for any pair of sets $A \subseteq B$. Notice that the half-product function $h(S)$ in general is not monotone. Moreover, throughout this paper we assume that a set-minimizer S_* and a set-maximizer S^* satisfy the conditions

$$h(S_*) < 0 = h(\emptyset) < h(S^*). \tag{4}$$

In terms of the set-functions $\varphi \in \{h, f\}$, the problems that we consider can be formulated as $\min\left\{\varphi(S)\,|S \in 2^N\right\}$ if no additional constraints are imposed, and as $\min\left\{\varphi(S)\,|\alpha(S) \le A,\ S \in 2^N\right\}$, if an additional knapsack constraint is introduced. The maximization counterparts of these problems are defined as $\max\left\{\varphi(S)\,|S \in 2^N\right\}$ and $\max\left\{\varphi(S)\,|\alpha(S) \le A,\ S \in 2^N\right\}$, respectively.

Problem $\min\left\{h(S)\,|S \in 2^N\right\}$ is NP-hard, as proved in [3]. The main focus of the paper is on design and analysis of approximation algorithms and schemes for the problems of minimizing half-product related functions.

We now recall the definitions related to approximation. In terms of set-functions, for a problem of minimizing a function $\varphi(S)$ which may take negative and positive values, a set S_ε is called an *ε-approximate solution* if for a given positive ε the inequality $\varphi(S_\varepsilon) - \varphi(S_*) \le \varepsilon\,|\varphi(S_*)|$ holds. A family of algorithms that for any given positive ε find an ε-approximate solution is called a *Fully Polynomial-Time Approximation Scheme (FPTAS),* provided that the running time depends polynomially on both the length of the input and $1/\varepsilon$.

The definitions introduced above are most traditional, since they measure the quality of an approximate solution in terms of relative errors. An alternative measure, which has received considerable attention is related to so-called *differential approximation*. Without going into technicalities, here we only mention that according to the differential approximation paradigm the quality of a solution $\varphi(S_H)$ is judged by its position in the interval $[\varphi(S_*), \varphi(S^*)]$. See [1] and [5] for details. In particular, for any positive ε a *Differential Fully Polynomial-Time Approximation Scheme (DFPTAS)* for minimizing a function $\varphi(S)$ delivers a feasible set S_ε such that $\varphi(S_\varepsilon) \le (1-\varepsilon)\,\varphi(S_*) + \varepsilon\varphi(S^*)$ and its running time depends polynomially on both the length of the input and $1/\varepsilon$.

There are several aspects that have provided motivations to this study.

1. The minimization problems $\min\left\{f(S)\,|S \in 2^N\right\}$ and $\min\left\{f(S)\,|\alpha(S) \le A,\ S \in 2^N\right\}$ serve as mathematical models of a range of scheduling problems, therefore approximation results available for these problems of quadratic Boolean programming can be adapted for various scheduling applications; see [6,17] as well as Section 4 of this paper.
2. Problem $\min\left\{h(S)\,|S \in 2^N\right\}$ admits a fast FPTAS due to [6], which cannot always be converted into an FPTAS for problem $\min\left\{f(S) = h(S) + K|S \in 2^N\right\}$ with an additive constant. However, handling additive constant is very easy for differential approximation algorithms.
3. Multiple results on differential approximation for various problems of combinatorial optimization are known, often producing a striking contrast with traditional approximation results. There is a lack of differential approximation results for scheduling problems. An exception is the paper [12], where a differential approximation algorithm (not a DFPTAS) is presented for the single machine problem of minimizing the weighted sum of the completion times subject to a single machine non-availability period. The latter problem is one of those which can be reformulated in terms of problem $\min\left\{f(S)\,|\alpha(S) \le A,\ S \in 2^N\right\}$. Thus, a natural question arises of deriving differential approximation results for problems similar to $\min\left\{f(S)\,|\alpha(S) \le A,\ S \in 2^N\right\}$ and therefore for the whole range of the relevant scheduling applications.

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