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On the polyhedra of graceful spheres and circular geodesics

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ABSTRACT

We construct a polyhedral surface called a *graceful surface*, which provides best possible approximation to a given sphere regarding certain criteria. In digital geometry terms, the graceful surface is uniquely characterized by its minimality while guaranteeing the connectivity of certain discrete (polyhedral) curves defined on it. The notion of "gracefulness" was first proposed in Brimkov and Barneva (1999) and shown to be useful for triangular mesh discretization through graceful planes and graceful lines. In this paper we extend the considerations to a nonlinear object such as a sphere. In particular, we investigate the properties of a discrete geodesic path between two voxels and show that discrete 3D circles, circular arcs, and Mobius triangles are all constructible on a graceful sphere, with guaranteed minimum thickness and the desired connectivity in the discrete topological space.

1. Introduction

In volume graphics, objects are represented by isothetic polyhedra composed by unit cubes (voxels) defined by the integer grid. In digital geometry for computer imagery and in combinatorial image analysis, such polyhedra are used to define discrete analogs of various Euclidean primitives (such as straight lines, planes, polygons, circles, or spheres), and obtaining an analytical description of a discrete object is highly desirable. In the present work, we introduce a new discrete model of a sphere and study its properties. Let us mention at this early point that at low-level image processing, digital objects and related properties and algorithms are usually represented and implemented in graph-theoretic terms, while in the scientific literature, for ease of understanding, these are traditionally given in terms of voxels and relations among them. In this paper, we conform to the latter paradigm.

As the sphere is a basic Euclidean primitive, a multitude of work on discrete spheres has been reported in the literature over the last two decades [1,3,5,19,11,15,16,14,21–23,27,35,34,36,42,43]. These are mostly of two types—one is on mathematical characterization of discrete spheres, and another is on voxelation techniques. Among those focused on voxelation, some are based on real specification and some on integer specification. In our work, we adopt an integer specification (i.e., integer radius and integer center) of the real sphere, which leads to an interesting and useful number-theoretic characterization of the resultant naive sphere, as recently shown in [6].

Another motivation for our study is found in possible applications in rapid prototyping (RP) technologies. With the advent of 3D imaging sciences and RP technologies, discrete spheres are widely used today for fabrication of different parts and products in diverse fields of science and technology, such as mechanical and electrical systems, microrobotics, biofabrication and bioprinting [18,25,26,28,30,31,33,41,45]. A series of work related to RP techniques in general, and 3D

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printing in particular, has come up over the last few years to address and strengthen the theoretical underpinnings in the digital technology. The 3D building matter is treated as digital in the sense that the building block is a digital unit or *voxel*, as opposed to analog (continuous) material used in conventional RP. Further, as the printing device works in discrete steps with a precision governed by the control system and its degrees of freedom, the input domain effectively maps to the integer space. As the physical size of a printable voxel can be as fine as a few microns, the actual radius of a 3D-printed sphere is obtained by specifying the integer radius in voxel unit [17,28,29,38].

The use of spheres in discrete and polyhedral form is also gradually becoming prevalent in many other areas of science and engineering where computational issues need to be addressed; to name a few, these are medical science, computational fluid dynamics and turbulence modeling, and optical science. For example, in medical science, discrete spheres have been used in [37,44] for detection of pulmonary nodules and for medical analysis of brains. For aircraft navigation, a sphere voxelation technique is used in [39,40] to compute the minimum distance between a set of points and a discrete surface element, and is shown to be more efficient than other alternatives for minimum distance computation. In the field of applied optics, polyhedral approximation of spheres has been used in [46] to circumscribe agglomerated debris particles for experimentation and study of light scattering in the process of discrete dipole approximation.

Note that, unlike in Euclidean geometry, having and using unique definition of a discrete object is a very rare event. In fact, a number of definitions exist for each of the basic Euclidean primitives, as usually any definition has advantages and disadvantages to the others depending on the specific context and applications. For example, in the framework of arithmetic geometry,[2] discrete straight lines and discrete planes are well-studied—and also quite well-established by today—for their deep theoretical connections to imaging sciences. Two important classes of discrete planes are the naive planes and the standard planes, which were mostly studied in the early stage [2,4,24,32]. Later, a new class of graceful planes was introduced in [8] to address the combinatorial issues related to generation of polyhedral meshes and 3D discrete polygons (triangles, for definiteness). 3D graceful lines were subsequently introduced as portions of graceful planes, which ensured that the sides of a 3D triangle are graceful, while the rest (interior) of the triangle remains naive.

The main objective of graceful discretization is to obtain simple analytical description (tunnel-free, in particular) of the basic Euclidean primitives and, on this basis, to create tools for efficient modeling of more sophisticated objects composed by such primitives [8]. Hence, alongside the studies related to naive spheres and standard spheres that are found in the literature of arithmetic geometry, a study on graceful spheres is imminent for a complete understanding of sphere discretization as in the case of plane discretization. With this motivation, we present here for the first time a study on the definition, characterization, and several discrete-geometric issues and outcomes related with graceful spheres. As a prelude, it is worth-mentioning that a graceful sphere in principle is analogous to a graceful plane by its fundamental property. Just like a graceful plane, which is the thinnest discretization of a Euclidean plane satisfying the minimal connectivity of any linear segment on it, a graceful sphere is also the thinnest discretization of a Euclidean sphere with the requisite property that any circular (geodesic) arc on it is a minimally connected set of voxels. The voxel set corresponding to the naive sphere becomes thinnest, and hence contains minimal number of voxels with 2-separating property. Since a graceful sphere is essentially built on a naive sphere with minimum addition of Steiner voxels, its voxel set eventually becomes thinnest and minimal too, and hence, in principle, conforms to the notion of a graceful surface explained in [8]. An introductory example showing a definite advantage of a graceful sphere over a naive sphere is shown in Fig. 1. As evident from this example, the connectivity of an edge (3D geodesic or circular arc) of a Mobius triangle is not guaranteed when defined on a naive sphere, which is however ensured when the edges are taken from the corresponding graceful sphere.

The rest of the paper is organized as follows. In Section 2, we discuss the preliminary ideas, definitions, metrics, and some topological basics related to different classes of discretization in the integer space. In Section 3, we introduce an analytical definition of graceful sphere, derive its related characterization, and explain how it ensures the connectivity of 3D circles and circular arcs, when defined on a graceful sphere in an appropriate topological framework. In Section 4, we establish the relation among naive, standard, and graceful spheres, and discuss different classes of algorithms for construction of Mobius triangles and resultant polygons with necessary connectivity on different types of discrete spherical surfaces. Finally, in Section 5, we draw our concluding notes with further scope and few open problems.

2. Preliminaries

2.1. Basic notions and notations

In this section, we fix some basic notions and notations to be used in the sequel. For more details we refer to [32]. Some other notions will be defined in the subsequent sections.

A *voxel* (also called a 3-cell) is a unit cube determined by the integer grid and fully identified by its center which is a point of $\mathbb{Z}^3$. A discrete (or digital) object is a finite set of voxels. The *supercover* of a set $M \subseteq \mathbb{R}^3$ is the set $S(M)$ of all voxels that are intersected by M.

Two voxels are said to be 0-*adjacent* if they share a vertex (0-cell), 1-*adjacent* if they share an edge (1-cell), and 2-*adjacent* if they share a face (2-cell). Thus, two distinct voxels, $p_1(i_1, j_1, k_1)$ and $p_2(i_2, j_2, k_2)$ are 1-*adjacent* if and only if

[2] also called "analytic discrete geometry".

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