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Threshold-coloring and unit-cube contact representation of planar graphs

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ABSTRACT

In this paper we study *threshold-coloring* of graphs, where the vertex colors represented by integers are used to describe any spanning subgraph of the given graph as follows. A pair of vertices with a small difference in their colors implies that the edge between them is present, while a pair of vertices with a big color difference implies that the edge is absent. Not all planar graphs are threshold-colorable, but several subclasses, such as trees, some planar grids, and planar graphs with no short cycles can always be threshold-colored. Using these results we obtain unit-cube contact representation of several subclasses of planar graphs. Variants of the threshold-coloring problem are related to well-known graph coloring and other graph-theoretic problems. Using these relations we show the NP-completeness for two of these variants, and describe a polynomial-time algorithm for another.

1. Introduction

Graph coloring is among the fundamental problems in graph theory. Typical applications of the problem and its generalizations are in job scheduling, channel assignments in wireless networks, register allocation in compiler optimization and many others [24]. In this paper[1] we consider a new graph coloring problem in which we assign colors (integers) to the vertices of a graph G in order to define a spanning subgraph H of G. In particular, we color the vertices of G so that for each edge of H, the two endpoints are near, that is, their difference is at most a given "threshold", and for each edge of $G \setminus H$, the endpoints are far, that is, their difference is greater than the threshold; see Fig. 1.

The motivation of the problem is severalfold. First, such coloring arises in the context of the geometric problem of *unit-cube contact representation* of planar graphs. In such a representation of a graph, each vertex is represented by a unit-size cube and each edge is realized by a common boundary with non-zero area between the two corresponding cubes. Finding classes of planar graphs with unit-cube contact representation was recently posed as an open question by Bremner et al. [5].

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1 A part of the results of this paper was presented at the *39th International Workshop on Graph-Theoretic Concepts in Computer Science* (WG'13) [1] and *7th International Conference on Fun With Algorithms* (FUN'14) [2].

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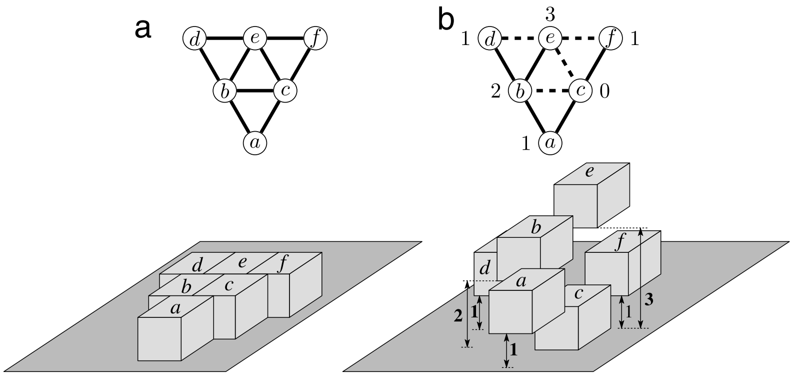

Fig. 1. (a) A planar graph $G = (V, E_G)$ and a corresponding unit-cube contact representation where the bottom faces of all cubes are co-planar, (b) a spanning subgraph $H = (V, E_H)$ of G with a (4, 1)-threshold-coloring for (G, H) and a corresponding unit-cube contact representation. The edges in $E_G \setminus E_H$ (far edges) are shown dashed, and the edges in E_H (near edges) are shown solid.

In this paper we partially address this problem as an application of our coloring problem in the following way. Suppose a planar graph G has a unit-cube contact representation where one face of each cube is co-planar; see Fig. 1(a). Assume that we can define a spanning subgraph H of G by our particular vertex coloring. We show that it is possible to compute a unit-cube contact representation of H by lifting the cube for each vertex v by the amount equal to the color of v (where the size or side-length of the cubes are roughly equal to the threshold); see Fig. 1(b).

Another motivation for the threshold-coloring comes from the notion of *adjacency labeling scheme* [6,14]. The idea is to label (color) vertices of a graph in a way that will allow one to infer the adjacency of two vertices directly from their labels without using additional information. Clearly labels of unrestricted size can be used to encode any desired information. However, for practical considerations, it is important to keep labels relatively short and allow efficient information deduction. Threshold-coloring makes it possible to determine the adjacency between two vertices of a graph in constant time. Finally, such coloring can be used for the Frequency Assignment Problem [18], which asks for assigning frequencies to transmitters in radio networks so that only specified pairs of transmitters can communicate with each other.

1.1. Problem definition

An *edge-labeling* of graph $G = (V, E)$ is a mapping $\ell : E \to \{N, F\}$ assigning labels N or F to each edge of the graph; we informally name edges labeled with N as the *near* edges, and edges labeled with F as the *far* edges. Note that such an edge-labeling of G defines a partition of the edges E into near and far edges. By abuse of notation the pair $\{N, F\}$ also denotes this partition.

Let $r \geq 1$ and $t \geq 0$ be two integers and let $[1 \dots r]$ denote a set of r consecutive integers. For a graph $G = (V, E)$ and an edge-labeling $\ell : E \to \{N, F\}$ of G, an (r, t)-*threshold-coloring* of G with respect to ℓ is a *coloring* $c : V \to [1 \dots r]$ such that for each edge $e = (u, v) \in E$, if $(u, v) \in N$ then $|c(u) - c(v)| \leq t$ and if $(u, v) \in F$ then $|c(u) - c(v)| > t$. We call r and t the *range* and the *threshold*, respectively. Note that the set of near edges defines a spanning subgraph $H = (V, N)$ of G, where H is a *spanning subgraph* of graph G if it contains all vertices of G. We can thus redefine threshold-coloring for a pair (G, H) of a graph $G = (V, E_G)$ and a spanning subgraph $H = (V, E_H)$ of G: an (r, t)-*threshold-coloring* for (G, H) is the one for G with respect to the labeling $\ell : E_G \to \{N, F\}$, where $\ell(e) = N$ if $e \in H$ and $\ell(e) = F$ if $e \notin H$. The graph H is a threshold-subgraph of G if there exists an (r, t)-threshold-coloring for (G, H) for some integers r, t.

A graph G is (r, t)-*total-threshold-colorable* for some $r \geq 1$, $t \geq 0$, if for every edge-labeling ℓ of G there exists an (r, t)-threshold-coloring of G with respect to ℓ. Informally speaking, for every partition of edges of G into near and far edges, we can produce vertex colors so that endpoints of near edges receive near colors, and endpoints of far edges receive colors that are far apart. A graph G is *total-threshold-colorable* if it is (r, t)-total-threshold-colorable for some range $r \geq 1$ and threshold $t \geq 0$. In this paper we focus on the following problem variants.

Total-Threshold-Coloring: Given a graph G, is G total-threshold-colorable, that is, is every spanning subgraph of G a threshold subgraph of G?

The problem is closely related to the question about whether a particular spanning graph H of G is threshold-colorable.

Threshold-Coloring: Given a graph G and a spanning subgraph H, is H a threshold subgraph of G for some integers $r \geq 1$, $t \geq 0$?

Another interesting variant of the threshold-coloring is the one in which we specify that the graph G is the complete graph. In this case we call H an *exact-threshold* graph if H is a threshold subgraph of the complete graph G for some integers $r \geq 1$, $t \geq 0$.

Exact-Threshold-Coloring: Given a graph H, is H an exact-threshold graph?

In the final variant of the problem we assume that the threshold and the range are the part of the input.