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Graphs with maximal induced matchings of the same size

Philippe Baptiste [a], Mikhail Y. Kovalyov [b], Yury L. Orlovich [c,*], Frank Werner [d], Igor E. Zverovich [e]

[a] Ecole Polytechnique, Laboratoire d'Informatique (CNRS LIX), F-91128 Palaiseau, France

[b] United Institute of Informatics Problems, National Academy of Sciences of Belarus, 6 Surganova Str., 220012 Minsk, Belarus

[c] Department of Discrete Mathematics and Algorithmics, Faculty of Applied Mathematics and Computer Science, Belarusian State University, Nezavisimosti Ave. 4, 220030 Minsk, Belarus

[d] Institute of Mathematical Optimization, Otto-von-Guericke-University of Magdeburg, Universitätsplatz 2, 39106 Magdeburg, Germany

[e] Department of Mathematical Cybernetics, Faculty of Mechanics and Mathematics, Belarusian State University, Nezavisimosti Ave. 4, 220030 Minsk, Belarus

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ABSTRACT

A graph is *well-indumatched* if all its maximal induced matchings are of the same size. We first prove that recognizing whether a graph is well-indumatched is a co-NP-complete problem even for $(2P_5, K_{1,5})$-free graphs. We then show that decision problems Independent Dominating Set, Independent Set, and Dominating Set are NP-complete for the class of well-indumatched graphs. We also show that this class is a co-indumatching hereditary class, i.e., it is closed under deleting the end-vertices of an induced matching along with their neighborhoods, and we characterize well-indumatched graphs in terms of forbidden co-indumatching subgraphs. We prove that recognizing a co-indumatching subgraph is an NP-complete problem. We introduce a *perfectly well-indumatched* graph, in which every induced subgraph is well-indumatched, and characterize the class of these graphs in terms of forbidden induced subgraphs. Finally, we show that the weighted versions of problems Independent Dominating Set and Independent Set can be solved in polynomial time for perfectly well-indumatched graphs, but problem Dominating Set is NP-complete.

1. Introduction

In this paper, we use graph-theoretic terminology of Bondy and Murty [7] (unless noted otherwise), and computational complexity terminology of Garey and Johnson [29].

A *matching* in a graph is a set of edges with no two edges having a common vertex. An *induced matching* is a matching with an additional property that no two of its edges are joined by an edge. An induced matching M in a graph G is *maximal* if no other induced matching in G contains M, while an induced matching of maximum size is a *maximum induced matching*. The problem of deciding if there exists an induced matching of a given size or larger is called Maximum Induced Matching. Induced matchings have applications in communication network testing [47], concurrent transmission of messages in

* Corresponding author.
E-mail addresses: philippe.baptiste@polytechnique.fr (P. Baptiste), kovalyov_my@newman.bas-net.by (M.Y. Kovalyov), orlovich@bsu.by (Y.L. Orlovich), frank.werner@mathematik.uni-magdeburg.de (F. Werner), zverovich@bsu.by (I.E. Zverovich).

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wireless ad hoc networks [2] and secure communication channels in broadcast networks [30]. The Maximum Induced Matching problem was also studied in [8–14,19,22,38,39,42,43], see [10,22] also for a survey and new results. In [39], Kobler and Rotics showed that the graphs where the sizes of a maximum matching and a maximum induced matching coincide, can be recognized in polynomial time. Later, Cameron and Walker [15] extended this result by giving a complete structural description of these graphs.

We call a graph *well-indumatched* if all its maximal induced matchings have the same size. For example, the graph obtained from a star $K_{1,n}$ by subdividing each edge by two vertices is well-indumatched. We denote by $\mathcal{WIM}$ the class of well-indumatched graphs. We study the problem of recognizing if a graph is well-indumatched, and examine computational complexity of problems Independent Dominating Set, Independent Set, and Dominating Set for such graphs.

The fact that any maximal induced matching of a well-indumatched graph has the maximum size implies that the problem Maximum Induced Matching can be solved by a greedy algorithm for well-indumatched graphs, or, in other words, class $\mathcal{WIM}$ forms a set of greedy instances in the sense of Caro et al. [18] with respect to the problem Maximum Induced Matching. Greedy instances of other combinatorial problems can be defined in a similar way (see, e.g., [18,23,32,40,50–52]).

The class of *well-covered* graphs is a class of greedy instances of the problem Independent Set [18]. A graph is well-covered if all its maximal independent sets are of the same size. This concept is introduced by Plummer [44], and applications exist in distributed computing systems [56]. The problem of recognizing well-covered graphs is proved to be co-NP-complete for general graphs, independently by Sankaranarayana and Stewart [46], and Chvátal and Slater [20]. It is co-NP-complete for $K_{1,4}$-free graphs [18], but solvable in polynomial time for $K_{1,3}$-free graphs [48,49]. Significant work has been done towards characterizing well-covered graphs such as trees and bipartite graphs (Ravindra [45]), graphs with a girth of at least 5 (Finbow et al. [24]), cubic graphs (Campbell et al. [16]), and plane triangulations (Finbow et al. [25–27]).

A graph is called *equimatchable* if all its maximal matchings have the same size, see Lovász and Plummer [41]. The class of equimatchable graphs and the class $\mathcal{WIM}$ of well-indumatched graphs can be viewed as edge analogues of well-covered graphs, because the property of maximal matchings (induced matchings) to have the same size correlates with that of all maximal independent vertex sets to have the same size. The problem of recognizing an equimatchable graph was first studied by Lesk et al. [40] who gave a characterization of equimatchable graphs in terms of Gallai–Edmonds structure theorem, see, e.g., [55], and showed that there exists a polynomial-time algorithm which decides whether a given graph is equimatchable.

In Section 2, we first show that recognizing a well-indumatched graph is a co-NP-complete problem even for $(2P_5, K_{1,5})$-free graphs. Then we prove that, for the same graphs, the problem of recognizing a graph that has maximal induced matchings of at most t distinct sizes is co-NP-complete for any given $t \geqslant 1$.

Let IMatch(G) be the set of all maximal induced matchings of the graph G. Define the *minimum maximal induced matching number* as $\sigma(G) = \min\{|M| : M \in \text{IMatch}(G)\}$, and the *maximum induced matching number* as $\Sigma(G) = \max\{|M| : M \in \text{IMatch}(G)\}$. Similar to the definition of a maximum induced matching that comprises $\Sigma(G)$ edges, we define a *min–max induced matching* as a maximal induced matching with $\sigma(G)$ edges. In a greedy way we can find both parameters $\sigma(G)$ and $\Sigma(G)$ in any well-indumatched graph G. It is well known that the problem of deciding "$\Sigma(G) \geqslant K$" is NP-complete [12,47]. Moreover, in general graphs with n vertices, the optimization version of the Maximum Induced Matching problem cannot be approximated within a factor of $n^{1/2-\varepsilon}$ for any constant $\varepsilon > 0$, unless P = NP [43]. In Section 2, we prove that the problem of deciding "$\sigma(G) \leqslant K$" is NP-complete even if the graph has maximal induced matchings of at most two sizes differing by one. Other results on the complexity and inapproximability of the problem associated with $\sigma(G)$ can be found in [43].

Ko and Shepherd [38] investigated relations between $\Sigma(G)$ and $\gamma(G)$, the domination number of a graph G. They write that they do not know any class of graphs for which exactly one of the values γ and Σ is computable in polynomial time. In Sections 3 and 5, we prove that problems Independent Dominating Set, Independent Set, and Dominating Set are NP-complete for well-indumatched graphs. Thus, for the class $\mathcal{WIM}$, calculating γ is an NP-hard problem (see Section 5), while Σ can be calculated in polynomial time by constructing an arbitrary maximal induced matching. Our proof for the Independent Set problem implies that the well-known problems Partition into Triangles, Chordal Graph Completion and Partition into Subgraphs Isomorphic to P_3 are NP-complete for well-indumatched graphs. NP-completeness of the latter problem implies that computing Σ is NP-hard even for Hamiltonian line graphs of well-indumatched graphs, which generalizes the result of Kobler and Rotics [39]. In Section 3, we also show that problems Graph k-Colorability and Clique are NP-complete for well-indumatched graphs.

A class of graphs is called *hereditary* if every induced subgraph of a graph in this class also belongs to the class. For a set $\mathcal{H}$ of graphs, a graph G is called $\mathcal{H}$*-free* if no induced subgraph of G is isomorphic to a graph in $\mathcal{H}$. In other words, $\mathcal{H}$-free graphs constitute a hereditary class defined by $\mathcal{H}$ as the set of forbidden induced subgraphs.

In Section 4, we observe that the class $\mathcal{WIM}$ of well-indumatched graphs is not hereditary, but we prove that $\mathcal{WIM}$ is a co-indumatching hereditary class, i.e., it is closed under deleting the end-vertices of an induced matching along with their neighborhoods. Also, in Section 4, we characterize well-indumatched graphs in terms of forbidden co-indumatching subgraphs. This means that we specify the minimal set $\mathcal{Z}_{\mathcal{WIM}}$ of graphs such that G is well-indumatched if and only if G does not contain any graph in $\mathcal{Z}_{\mathcal{WIM}}$ as a co-indumatching subgraph. We prove that recognizing a co-indumatching subgraph is an NP-complete problem. We conclude Section 4 by presenting a variant of the well-known result of Chvátal and Slater [20] and Sankaranarayana and Stewart [46] that recognizing well-covered graphs is co-NP-complete. Namely, we show that recognizing well-covered graphs is co-NP-complete even for $(2P_5, K_{1,10})$-free squares of line graphs.