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Characterizing width two for variants of treewidth[☆]

Hans L. Bodlaender[a,b], Stefan Kratsch[c], Vincent J.C. Kreuzen[d], O-joung Kwon[e,*], Seongmin Ok[f]

[a] Department of Computing Science, Utrecht University, P.O. Box 80.089, 3508 TB, Utrecht, The Netherlands

[b] Department of Mathematics and Computer Science, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

[c] Department for Computer Science I, University of Bonn, Germany

[d] School of Business and Economics, Quantitative Economics, Maastricht University, The Netherlands

[e] Department of Mathematical Sciences, KAIST, 291 Daehak-ro Yuseong-gu Daejeon, 305-701, South Korea

[f] DTU Compute, Technical University of Denmark, DK-2800 Lyngby, Denmark

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ABSTRACT

In this paper, we consider the notion of *special treewidth*, recently introduced by Courcelle (2012). In a special tree decomposition, for each vertex v in a given graph, the bags containing v form a rooted path. We show that the class of graphs of special treewidth at most two is closed under taking minors, and give the complete list of the six minor obstructions. As an intermediate result, we prove that every connected graph of special treewidth at most two can be constructed by arranging blocks of special treewidth at most two in a specific tree-like fashion.

Inspired by the notion of special treewidth, we introduce three natural variants of treewidth, namely *spaghetti treewidth*, *strongly chordal treewidth* and *directed spaghetti treewidth*. All these parameters lie between pathwidth and treewidth, and we provide common structural properties on these parameters. For each parameter, we prove that the class of graphs having the parameter at most two is minor closed, and we characterize those classes in terms of a *tree of cycles* with additional conditions. Finally, we show that for each $k \geq 3$, the class of graphs with special treewidth, spaghetti treewidth, directed spaghetti treewidth, or strongly chordal treewidth, respectively at most k, is not closed under taking minors.

1. Introduction

Treewidth and *pathwidth* are one of the basic parameters in graph algorithms and they play an important role in structural graph theory. Numerous problems which are NP-hard on general graphs, have been shown to be solvable in polynomial time on graphs of bounded treewidth [2,6]. Courcelle [14] provided a celebrated algorithmic meta-theorem which states that every graph property expressible in monadic second-order logic formulas (MSO_2) can be decided in linear time on graphs of bounded treewidth.

In this paper, we discuss a relatively new notion of *special treewidth*, introduced by Courcelle [15]. A special tree decomposition is a tree decomposition where for each vertex of a given graph, the bags containing this vertex form a rooted path

☆ An earlier paper on the characterization of special treewidth two appeared in the proceedings of WG 2013 (Bodlaender et al., 2013).

* Corresponding author.
E-mail addresses: h.l.bodlaender@uu.nl (H.L. Bodlaender), kratsch@cs.uni-bonn.de (S. Kratsch), v.kreuzen@maastrichtuniversity.nl (V.J.C. Kreuzen), ojoung@kaist.ac.kr (O.-j. Kwon), seok@dtu.dk (S. Ok).

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Table 1
Graph parameters which can be defined by the clique number of a supergraph from a class of graphs. Graph classes are defined in Section 2.

Parameter	Graph class
Treewidth	Chordal graphs
Pathwidth	Interval graphs
Special treewidth	RDV graphs
Directed spaghetti treewidth	DV graphs
Spaghetti treewidth	UV graphs
Strongly chordal treewidth	Strongly chordal graphs
Treedepth	Trivially perfect graphs

in the tree. Courcelle developed this parameter to reduce the difficulty in representing tree decompositions algebraically. The monadic second-order logic (MSO_2) checking algorithm for treewidth in the meta-theorem is based on the constructions of finite automata, and he observed that these constructions become much simpler when working with special tree decompositions compared to standard tree decompositions.

Courcelle asked several questions on properties of special treewidth. One of the questions was how to characterize the class of graphs of special treewidth at most k by forbidden configurations. In this context, he showed that the graphs of special treewidth one are exactly the forests, but if $k \geq 5$, then the class of graphs of special treewidth at most k is not closed under taking minors.

In this paper, we prove that the class of graphs of special treewidth at most two is closed under taking minors, and provide the minor obstruction set. We also sharpen Courcelle's bound, and show that for $k \geq 3$, the class of graphs of special treewidth at most k is not closed under taking minors. The graph K_4 denotes the complete graph on four vertices, and the other five graphs are depicted in Figs. 1 and 2.

Theorem 5.8. *A graph has special treewidth at most two if and only if it has no minor isomorphic to* K_4, D_3, S_3, G_1, G_2*, or* G_3.

To show this, we first prove that every block of special treewidth at most two must have pathwidth at most two. But it is not a sufficient condition for having special treewidth two, and we establish a precise condition how those blocks can be attached to obtain a graph of special treewidth two.

Inspired by special treewidth, we introduce new three variants of treewidth. From the results by Courcelle, we observe that having bounded special treewidth is a much stronger property than having bounded treewidth. We can naturally ask whether there exist elegant width parameters lying between special treewidth and treewidth, which establish a link from pathwidth to treewidth.

Two variants, *spaghetti treewidth* and *directed spaghetti treewidth*, are defined by taking different models of tree decompositions. While in the intersection model of special treewidth, we associate each vertex with a rooted path, in a spaghetti tree decomposition, the bags containing each vertex form a 'usual' path in a tree (that is, without the condition of being rooted), and in a directed spaghetti tree decomposition, the bags containing each vertex form a directed path in a tree with a given direction. The *strongly chordal treewidth* of a graph G is defined as the minimum of the clique number of H minus one over all strongly chordal supergraphs H of G. These parameters are at most the pathwidth and at least the treewidth of the graph.

Each of these new parameters can be alternatively defined as the minimum of the clique number of all supergraphs where the supergraphs belong to a certain graph class. Another related notion is treedepth [8,9], and it can be defined as the minimum of the clique number of all trivially perfect supergraphs of a given graph. Table 1 gives an overview of the parameters and the corresponding classes.

We expect that these new parameters can be used to provide a link between pathwidth and treewidth by yielding new structural or algorithmic results. As a similar approach, Fomin, Fraigniaud, and Nisse [22] introduced a parameterized variant of tree decompositions, called *q-branched tree decompositions*, and provided a unified method to compute pathwidth and treewidth. In this paper, we study common structural properties of our notions.

For each of the three parameters, we show that the class of graphs having width at most two is closed under taking minors. Moreover, we precisely describe how those graphs look like in terms of *trees of cycles* with specific conditions depending on the parameter. Trees of cycles were used to characterize treewidth two [10] and pathwidth two [7]. In Table 2, we see an overview of the different parameters and the minor obstruction sets for these classes. As 2-connected graphs play a special role in several proofs, the 2-connected graphs in the obstruction sets are given in the second column. In addition, for each of these parameters and each value $k \geq 3$, we show that the class of graphs with the parameter at most k is not closed under taking minors.

Our characterizations in terms of forbidden minors fit in a line of research, originated by the ground breaking results in the graph minor project by Robertson and Seymour [33]. From the results of Robertson and Seymour, for every minor-closed class $\mathcal{G}$ of graphs, there exists a finite obstruction set $ob(\mathcal{G})$ of graphs such that for each graph H, $H \in \mathcal{G}$ if and only if H has no minor isomorphic to a graph in $ob(\mathcal{G})$. For several minor-closed graph classes, the obstruction set is known, for example, planar graphs ($\{K_5, K_{3,3}\}$ [37]), graphs embeddable in the projective plane [1], graphs of treewidth at most two ($\{K_4\}$, see [17, Proposition 12.4.2]), graphs of treewidth at most three (a set of four graphs [3]), graphs of pathwidth at most two (a

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