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Forbidden induced subgraphs of normal Helly circular-arc graphs: Characterization and detection[☆]

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ABSTRACT

A normal Helly circular-arc graph is the intersection graph of a set of arcs on a circle of which no three or less arcs cover the whole circle. Lin et al. (2013) characterized circular-arc graphs that are not normal Helly circular-arc graphs, and used them to develop the first recognition algorithm for this graph class. As open problems, they ask for the forbidden subgraph characterization and a direct recognition algorithm for normal Helly circular-arc graphs, both of which are resolved by the current paper. Moreover, when the input is not a normal Helly circular-arc graph, our recognition algorithm finds in linear time a minimal forbidden induced subgraph as a certificate. Our approach yields also a considerably simpler algorithm for the certifying recognition of proper Helly circular-arc graphs, a subclass of normal Helly circular-arc graphs.

1. Introduction

A graph is a *circular-arc graph* if its vertices can be assigned to arcs on a circle such that two vertices are adjacent if and only if their corresponding arcs intersect. Such a set of arcs is called a *circular-arc model* for this graph. If some point on the circle is not covered by any arc in the model, then the graph can also be represented by a set of intervals on the real line. This set of intervals is called an *interval model*, and the graph is an *interval graph*. The intersection graph of a set of subtrees of a tree is called a *chordal graph*. Circular-arc graphs, interval graphs, and chordal graphs are three of the most famous intersection graph classes, and have been studied intensively for more than half century. In contrast to interval graphs and chordal graphs, however, our understanding of circular-arc graphs is far limited, and to date some fundamental problems remain unsolved.

One fundamental combinatorial problem on a graph class is its characterization by forbidden (induced) subgraphs. For example, the forbidden induced subgraphs of *chordal graphs* are holes (i.e., induced cycles of length at least four). Lekkerkerker and Boland [13] showed in 1962 that the forbidden induced subgraphs of interval graphs include holes and graphs in Fig. 1. In contrast, the characterization of circular-arc graphs by forbidden induced subgraphs remains a notorious open problem since it was first asked by Hadwiger et al. [7] in 1964, though previous attempts did have partial success, which is mainly on subclasses of circular-arc graphs. For example, Tucker [23] characterized unit circular-arc graphs (i.e., a graph with a circular-arc model where every arc has the same length) and proper circular-arc graphs (i.e., a graph with a circular-arc model where

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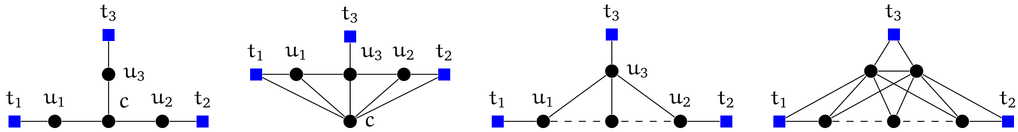
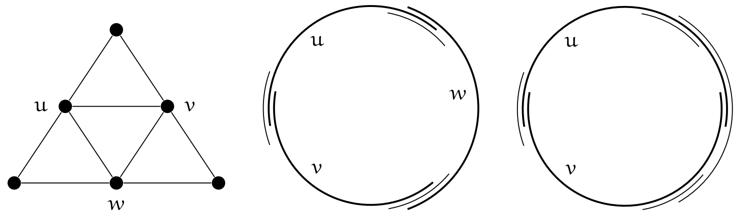


(a) Long claw. (b) Whipping top. (c) †. (d) ‡.

Fig. 1. Chordal minimal forbidden induced graphs.



(a) Tent (a ‡ on 6 vertices). (b) A normal model. (c) A Helly model.

Fig. 2. Tent and its circular-arc models. The arcs ($\{u, v, w\}$) invalidating the Helly property in (b) and the arcs ($\{u, v\}$) invalidating the normal property in (c) are marked as thick.

no arc properly contains another). We refer to the surveys of Lin and Soulignac [17] and of Durán et al. [4] for recent results in this line. The recent breakthrough of Francis et al. [5] may shed some light on the final resolution of this problem.

One fundamental algorithmic problem on a graph class is its recognition, i.e., to efficiently decide whether a given graph belongs to this class or not. For intersection graph classes, all recognition algorithms known to the authors provide an intersection model or some equivalent structure when the membership is asserted. Most of them, on the other hand, simply return "NO" for non-membership, while one might also want a verifiable *certificate* for some reason [19]. A recognition algorithm is *certifying* if it provides both positive and negative certificates. There are different forms of negative certificates, while a minimal forbidden induced subgraph is arguably the simplest and most preferable of them [8]. For example, it is long known that a hole can be detected from a non-chordal graph in linear time [22]. Very recently, Lindzey and McConnell [14] reported a linear-time algorithm that detects a subgraph in Fig. 1 from a chordal non-interval graph. They together make a linear-time certifying algorithm for the recognition of interval graphs. On the other hand, although a circular-arc model for a circular-arc graph can be produced in linear time [18], it remains a challenging open problem to find a negative certificate for a non-circular-arc graph in the same time.

The complication of circular-arc graphs may be attributed to two special intersection patterns of circular-arc models that are not possible in interval models. The first is two arcs intersecting in their both ends, and a circular-arc model is called *normal* if no such pair exists. The second is a set of arcs intersecting pairwise but containing no common point, and a circular-arc model is called *Helly* if no such set exists. Normal and Helly circular-arc models are precisely those without three or less arcs covering the whole circle [20,15]. A graph that admits such a model is called a *normal Helly circular-arc graph*. Clearly, all interval graphs are normal Helly circular-arc graphs; indeed, one may verify that all normal Helly circular-arc graphs that are chordal are interval graphs.

A word of caution is worth on the definition of normal Helly circular-arc graphs. One graph might admit both a normal circular-arc model and a Helly circular-arc model but not a circular-arc model that is both normal and Helly. See Fig. 2 for an example. One may want to verify that arranging a normal and Helly circular-arc model for a tent (i.e., a ‡ on 6 vertices) is out of the question. This example convinces us that the class of normal Helly circular-arc graphs is *not* equivalent to the intersection of the class of normal circular-arc graphs and the class of Helly circular-arc graphs, but a proper subset of it.

Let us mention some previous work related to normal Helly circular-arc graphs. The algorithm of Tucker [24] colors a normal Helly circular-arc graph using at most $3\omega/2$ colors, where ω denotes the size of its maximum cliques. Note that by the Helly property, ω is equivalent to the maximum number of arcs covering a single point on the circle. This is tight as any odd hole, which has $\omega = 2$ and needs at least three colors, is a normal Helly circular-arc graph. In the study of convergence of circular-arc graphs under the clique operator, Lin et al. [16] observed that normal Helly circular-arc graphs arose naturally. They [15] then undertook a systematic study of normal Helly circular-arc graphs as well as its subclass. Their results include a partial characterization of normal Helly circular-arc graphs by forbidden induced subgraphs (more specifically, those restricted to Helly circular-arc graphs), and a linear-time recognition algorithm (by calling a recognition algorithm for circular-arc graphs). As open problems, they ask for determining the remaining minimal forbidden induced subgraphs, and designing a direct recognition algorithm, both of which are resolved by the current paper.

The first main result of this paper is a complete characterization of normal Helly circular-arc graphs by forbidden induced subgraphs. A wheel (resp., C^*) comprises a hole and another vertex completely adjacent (resp., nonadjacent) to the hole.

Theorem 1.1. *A graph is a normal Helly circular-arc graph if and only if it contains no C^*, wheel, or any graph depicted in* Figs. 1 and 3 *as an induced subgraph.*