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Rainbow colouring of split graphs

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ABSTRACT

A *rainbow path* in an edge coloured graph is a path in which no two edges are coloured the same. A *rainbow colouring* of a connected graph G is a colouring of the edges of G such that every pair of vertices in G is connected by at least one rainbow path. The minimum number of colours required to rainbow colour G is called its *rainbow connection number*. It is known that, unless P = NP, the rainbow connection number of a graph cannot be approximated in polynomial time to a multiplicative factor less than 5/4, even when the input graph is chordal [Chandran and Rajendraprasad, FSTTCS 2013]. In this article, we determine the computational complexity of the above problem on successively more restricted graph classes, viz.: split graphs and threshold graphs. In particular, we establish the following:

1. The problem of deciding whether a given split graph can be rainbow coloured using k colours is NP-complete for $k \in \{2, 3\}$, but can be solved in polynomial time for all other values of k. Furthermore, any split graph can be rainbow coloured in linear time using at most one more colour than the optimum.
2. For every positive integer k, threshold graphs with rainbow connection number k can be characterised based on their degree sequence alone. Furthermore, we can optimally rainbow colour a threshold graph in linear time.

1. Introduction

An *edge colouring* of a graph is a function from its edge set to the set of natural numbers. A path in an edge coloured graph with no two edges sharing the same colour is called a *rainbow path*. An edge coloured graph is said to be *rainbow connected* if every pair of vertices is connected by at least one rainbow path. Such a colouring is called a *rainbow colouring* of the graph. A rainbow colouring using minimum possible number of colours is called *optimal*. The minimum number of colours required to rainbow colour a connected graph G is called its *rainbow connection number*, denoted by $rc(G)$. For example, the rainbow connection number of a complete graph is 1, that of a path is its length, that of an even cycle is half its length, and that of a tree is its number of edges. Note that disconnected graphs cannot be rainbow coloured and hence their rainbow connection number is left undefined. Any connected graph can be rainbow coloured by giving distinct colours to the edges of a spanning tree of the graph. Hence the rainbow connection number of any connected graph is less than its number of

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vertices. It is trivial to see that $rc(G)$ is at least the diameter of G. It is easy to see that no two bridges in a graph can get the same colour under a rainbow colouring and hence $rc(G)$ is lower bounded by the number of bridges in the G.

The concept of rainbow colouring was introduced by Chartrand, Johns, McKeon, and Zhang in [9] where they also determined the precise values of rainbow connection number for some special graphs. Subsequently, there have been various investigations towards finding good upper bounds for rainbow connection number in terms of other graph parameters [4,22,6,2] and for many special graph classes [20,6,3]. Behaviour of rainbow connection number in random graphs is also well studied [4,16,23,13]. A basic introduction to the topic can be found in Chapter 11 of the book *Chromatic Graph Theory* by Chartrand and Zhang [10] and a survey of most of the recent results in the area can be found in the article by Li and Sun [19] and also in their monograph *Rainbow Connection of Graphs* [21].

In this article, we focus on the computational complexity of the following decision problem.

Problem 1 (RainbowColour(G, k))**.** Given a connected graph G and a positive integer k, decide whether G can be rainbow coloured using k colours.

The first result showing the computational complexity of the above problem was due to Chakraborty, Fischer, Matsliah, and Yuster [5]. They showed that it is NP-hard to compute the rainbow connection number of an arbitrary graph. In particular, it was shown that the problem RainbowColour(G, 2) is NP-complete. Later, Ananth, Nasre, and Sarpatwar [1] complemented the above result and now we know that for every integer k, $k \geq 2$, the problem RainbowColour(G, k) is NP-complete. This prompts one to look at the computational complexity of the problem on special graph classes. Chandran and Rajendraprasad have established that, for every positive integer k, it is NP-hard to distinguish between chordal graphs with rainbow connection number $4k$ and $5k$ [8]. Hence there cannot exist a polynomial time algorithm to rainbow colour chordal graphs with less than 5/4 times the optimum number of colours, unless P = NP. Here, we determine the computational complexity of the above problem on successively more restricted graph classes, viz.: split graphs and threshold graphs. Before stating our results, we formally define the above graph classes and introduce some fairly standard notation.

1.1. Notations, definitions and preliminaries

All graphs considered in this article are finite, simple and undirected. Since disconnected graphs cannot be rainbow coloured, we assume that all the graphs discussed here are connected, unless mentioned otherwise. For a graph G, we use $V(G)$ and $E(G)$ to denote its vertex set and edge set respectively. An edge $\{u, v\}$ in a graph may be denoted by uv to reduce clutter. Unless mentioned otherwise, n and m will respectively denote the number of vertices and edges of the graph in consideration. The subgraph of G induced on a vertex set $S \subset V(G)$ is denoted by $G[S]$. The *length* of a path is its number of edges. The *distance* between two vertices u and v in a graph G, denoted by $d(u, v)$ is the length of a shortest path between them in G. The *diameter* of a connected graph G is $diam(G) = \max_{u,v \in V(G)} d(u, v)$. The *neighbourhood* $N(v)$ of a vertex v is the set of vertices adjacent to v. The *degree* of a vertex v is $d_v = |N(v)|$. The *degree sequence* of a graph is the non-increasing sequence of its vertex degrees. A vertex is called *pendant* if its degree is 1. An edge incident on a pendant vertex is called a *pendant edge* and the set of pendant vertices of a graph G is denoted by $\text{pen}(G)$. The shorthand $[n]$ denotes the set $\{1, \dots, n\}$. The cardinality of a set S is denoted by $|S|$ and the family of all subsets of S is denoted by 2^S. The union of two disjoint sets A and B is denoted by $A \dot{\cup} B$.

Definition 1 (*Graph Classes*)**.** A graph G is called *chordal*, if there is no induced cycle of length greater than 3. A graph G is a *split graph*, if $V(G)$ can be partitioned into a clique and an independent set. A graph G is a *threshold graph*, if there exists a weight function $w : V(G) \to \mathbb{R}$ and a real constant t such that two vertices $u, v \in V(G)$ are adjacent if and only if $w(u) + w(v) \geq t$.

It is well known and easy to verify that threshold graphs are a subclass of split graphs which in turn are a subclass of chordal graphs. We also note that we can find a maximum clique and a maximum independent set in a split graph (and hence a threshold graph) in linear time as follows. The vertices of a graph can be sorted according to their degrees in $O(n)$ time using a counting sort [15]. If $G([n], E)$ is a split graph with the vertices labelled so that $d_1 \geq \cdots \geq d_n$, where d_i is degree of vertex i, then $\{i \in V(G) : d_i \geq i - 1\}$ is a maximum clique in G and $\{i \in V(G) : d_i \leq i - 1\}$ is a maximum independent set in G [14]. Hence we can assume, if needed, that a maximum clique or a maximum independent set or an ordering of the vertices according to their degrees is given as input to our algorithms.

A *hypergraph* H is a tuple (V, E), where V is a finite set and $E \subseteq 2^V$. Elements of V and E are called vertices and (hyper-)edges respectively. The hypergraph H is called *r-uniform* if $|e| = r$ for every $e \in E$. An r-uniform hypergraph is called *complete* if $E = \{e \subset V : |e| = r\}$.

Definition 2 (*Hypergraph Colouring*)**.** Given a hypergraph $H(V, E)$ and a colouring $C_H : V \to \mathbb{N}$, an edge is called *k-coloured* if the edge contains vertices of k different colours. An edge is called *monochromatic* if it is 1-coloured. The colouring C_H is called *proper* if no edge in E is monochromatic under C_H. The minimum number of colours required to properly colour H is called its *chromatic number* and is denoted by $\chi(H)$.