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On neighborhood-Helly graphs

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ABSTRACT

A family $\mathcal{F}$ of subsets of some set is *intersecting* when sets of $\mathcal{F}$ pairwise intersect. The family $\mathcal{F}$ is *Helly* when every intersecting subfamily of it contains a common element. In this paper we examine the families of vertex neighborhoods of a graph, with the aim of determining whether or not they are Helly, and also whether or nor they are *hereditary Helly*, that is, each of the induced subgraphs of the graph is Helly. We examine the cases where the neighborhoods are all open, or all closed, or mixed, that is, some open and some closed. For mixed neighborhoods there are two different kinds of choice of the neighborhood of each vertex to be considered: fixed or arbitrary choice. By fixed mixed neighborhood, we mean that the choice, open or closed, for the neighborhood of a vertex is known in advance, that is part of the input. On the other hand, an arbitrary choice implies that the choice can be made along the process. For the cases of open, closed and fixed mixed neighborhoods, we describe characterizations, both for the neighborhoods to be Helly and hereditary Helly. The characterizations are of two types: based on the concept of extensions, or, for the hereditary cases, by forbidden induced subgraphs. Polynomial time recognition algorithms follow directly from the characterizations. In contrast, for arbitrary mixed neighborhoods, we prove that it is NP-complete to decide whether the family of neighborhoods is Helly or hereditary Helly.

1. Introduction

Denote by G a finite simple graph, with vertex set $V(G)$ and edge set $E(G)$. We use n and m to denote $|V(G)|$ and $|E(G)|$. A *complete set* is a subset $V' \subseteq V(G)$ formed by pairwise adjacent vertices. A *triangle* is a complete set of size 3 and a subset of vertices is a *co-triangle* when it is a triangle in $\overline{G}$, the complement of G. Denote by $N(v_i) = \{v_j \in V(G) | (v_i, v_j) \in E(G)\}$, and $N[v_i] = N(v_i) \cup \{v_i\}$, the *open* and *closed neighborhoods* of G, respectively. The degree of a vertex v_i, $d(v_i)$, is $|N(v_i)|$ and the maximum degree of G is denoted by Δ. For $V' \subseteq V(G)$, $G[V']$ is the subgraph of G induced by V'. Let $\mathcal{F}$ be a family of subsets of some set. Say that $\mathcal{F}$ is *intersecting* when the subsets of $\mathcal{F}$ pairwise intersect. On the other hand, when every intersecting subfamily of $\mathcal{F}$ has a common element then $\mathcal{F}$ is a *Helly* family.

The Helly property in the context of graphs and hypergraphs has been considered in many papers. Among them, we can mention [1–23,26,24,25,27–31,33–36].

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A graph G is *open neighborhood-Helly (closed neighborhood-Helly)* when its family of open neighborhoods (closed neighborhoods) is Helly. Finally, G is *hereditary open neighborhood-Helly (hereditary closed neighborhood-Helly)* when every of its induced subgraphs is open neighborhood-Helly (closed neighborhood-Helly).

Different characterizations were given for these graph classes and most of them lead to polynomial-time recognition algorithms: closed neighborhood-Helly graphs [15,24]; open neighborhood-Helly graphs [24]; hereditary closed neighborhood-Helly graphs [25]; hereditary open neighborhood-Helly graphs [25].

A natural extension of these classes is the case where the neighborhoods are not necessarily all open or all closed, that is they are of some mixed type. In this situation, there is a partition of the vertices, according how each neighborhood is to be considered, open or closed. Such a partition can be fixed or variable. Say that G is *fixed mixed neighborhood-Helly* when for a given partition of its vertices, into open and closed, the corresponding neighborhoods satisfy the Helly property. On the other hand, G is *arbitrary mixed neighborhood-Helly* when there exists some partition of the vertices which turns the neighborhoods to be Helly. Accordingly, define the concepts of *fixed hereditary mixed neighborhood-Helly* and *arbitrary hereditary mixed neighborhood-Helly*.

In this work, we describe characterizations, based on the concept of extensions, for the classes of neighborhood-Helly graphs: open, closed and mixed, both fixed and arbitrary. In addition, we also describe characterizations for their corresponding hereditary classes. As a consequence, we can obtain naturally polynomial time recognition algorithms for the classes into consideration, except for two cases. The exceptions are those of the arbitrary mixed classes. We prove that it is NP-complete to recognize arbitrary mixed neighborhood-Helly graphs and arbitrary hereditary mixed neighborhood-Helly graphs. The NP-completeness remains even for graphs of maximum degree 4.

We describe two different types of characterizations, those based on the concept of *extensions*, and, for hereditary classes, forbidden subgraph characterizations.

In Section 2, we present some terminology relevant to this work, open neighborhoods and closed neighborhoods are respectively considered in Sections 3 and 4, whereas fixed and arbitrary mixed neighborhoods are the subjects of Sections 5 and 6, respectively.

2. Preliminaries

First, we describe additional notation and definitions.

In the case of mixed neighborhoods, we may employ the notation $N\{v\}$ to mean that the neighborhood to be considered for vertex v, open or closed, is not determined.

Denote by $N[v, w]$ the intersection of $N[v]$ and $N[w]$, i.e. $N[v, w] = N[v] \cap N[w]$. On the other hand, we define the universal set of v, w as: $U[v, w] = \{u \in V(G)/N[v, w] \subseteq N[u]\}$. The universal set of u, v, w is defined as: $U[u, v, w] = U[u, v] \cap U[v, w] \cap U[u, w]$.

Define the *extension* of vertices u, v, w, as $E(u, v, w) = N[u, v] \cup N[v, w] \cup N[u, w]$, whenever $N[u, v]$, $N[v, w], N[u, w] \neq \emptyset$, and $E(u, v, w) = \emptyset$ otherwise.

A *chord* of cycle C is an edge of the graph between two vertices v_i, v_j not consecutive in C. If v_i, v_j are at distance two in C then the edge v_iv_j is called a *short chord*.

There is a polynomial-time general algorithm to check the Helly property for a family of subsets of polynomial size, based on the following characterization.

Theorem 2.1 ([5]). *Let $\mathcal{F}$ be a family of subsets of some set U. Given three different elements $u, v, w \in U$, let $\mathcal{F}_{\{u,v,w\}}$ be the subfamily of $\mathcal{F}$ formed by the sets containing at least two of these three elements. $\mathcal{F}$ is Helly if and only if for every triple $\{u, v, w\} \subseteq U$, $\bigcap S \neq \emptyset$, where $S \in \mathcal{F}_{\{u,v,w\}}$.*

3. Open neighborhood

Open neighborhood-Helly graphs have been characterized in terms of extensions.

Theorem 3.1 ([24]). *A graph G is open neighborhood-Helly if and only if it does not contain triangles, and for every 3-independent set {u,v,w} there exists a vertex $z \in V(G)$ such that the extension $E(u, v, w) \subseteq N[z]$, i.e, $U(u, v, w) \neq \emptyset$ or $E(u, v, w) = \emptyset$.*

For hereditary open neighborhood-Helly graphs, we describe two characterizations. The first is by forbidden induced subgraphs.

Theorem 3.2 ([25]). *Let G be a graph. Then G is hereditary open neighborhood-Helly if and only if G does not contain C_6 nor triangles as induced subgraphs.*

Next, we formulate a characterization of hereditary open neighborhood-Helly graphs in terms of extensions.

Theorem 3.3. *A graph G is hereditary open neighborhood-Helly if and only if it does not contain triangles and for every triple of pairwise non-adjacent vertices u, v, w such that $N[u, v], N[v, w], N[u, w] \neq \emptyset$ then either $E(u, v, w) \subseteq N[u]$, $E(u, v, w) \subseteq N[v]$ or $E(u, v, w) \subseteq N[w]$, i.e. $U(u, v, w) \cap \{u, v, w\} \neq \emptyset$ or $E(u, v, w) = \emptyset$.*