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Graph isomorphism for graph classes characterized by two forbidden induced subgraphs[☆]

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ABSTRACT

We study the complexity of the Graph Isomorphism problem on graph classes that are characterized by a finite number of forbidden induced subgraphs, focusing mostly on the case of two forbidden subgraphs. We show hardness results and develop techniques for the structural analysis of such graph classes. Applied to the case of two forbidden subgraphs we obtain the following main result: A dichotomy into isomorphism complete and polynomial-time solvable graph classes for all but finitely many cases, whenever neither forbidden graph is a clique, a pan, or a complement of these graphs. Further reducing the remaining open cases we show that (with respect to graph isomorphism) forbidding a pan is equivalent to forbidding a clique of size three.

1. Introduction

Given two graphs G_1 and G_2, the Graph Isomorphism problem (GI) asks whether there exists a bijection from the vertices of G_1 to the vertices of G_2 that preserves adjacency. This paper studies the complexity of GI on graph classes that are characterized by a finite number of forbidden induced subgraphs, focusing mostly on the case of two forbidden subgraphs. For a set of graphs $\{H_1, \ldots, H_k\}$ we let $(H_1, \ldots, H_k)$-free denote the class of graphs G that do not contain any H_i as an induced subgraph.

As a first example, consider the class of graphs containing neither a clique K_s on s vertices, nor an independent set I_t on t vertices. Ramsey's Theorem [19] states that the number of vertices in such graphs is bounded by a function $f(s, t)$. Thus the classes (K_s, I_t)-free are finite and Graph Isomorphism is trivial on them. All other combinations of two forbidden subgraphs give graph classes of infinite size, since they contain infinitely many cliques or independent sets.

As a second example, consider the graphs containing no clique K_s on s vertices and no star $K_{1,t}$ (i.e., an independent set of size t with added universal vertex adjacent to every other vertex). On the one hand this class contains all graphs of maximum degree less than $\min\{s-1, t\}$, on the other hand, all graphs in $(K_s, K_{1,t})$-free have bounded degree: Indeed, if the degree of a vertex is sufficiently large, its neighborhood must contain a clique of size s or an independent set of size t by Ramsey's Theorem [19], leading to one of the two forbidden subgraphs. Thus, using Luks' algorithm [16] that solves Graph Isomorphism on graphs of bounded degree in polynomial time, isomorphism of $(K_s, K_{1,t})$-free graphs can also be decided in polynomial time.

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To systematically study Graph Isomorphism on graph classes characterized by forbidden subgraphs, we ask: *Given a set of graphs* $\{H_1, \ldots, H_k\}$, *what is the complexity of Graph Isomorphism on the class of* $(H_1, \ldots, H_k)$-free *graphs?*

Related work. The Graph Isomorphism problem is contained in the complexity class NP, since the adjacency preserving bijection (the isomorphism) can be checked in polynomial time. No polynomial-time algorithm is known and it is known that Graph Isomorphism is not NP-complete unless the polynomial hierarchy collapses [5]. More strongly, Graph Isomorphism is in the low hierarchy [21]. This has led to the definition of the complexity class of problems polynomially equivalent to Graph Isomorphism, the so-called GI-complete problems. There is a vast literature on the Graph Isomorphism problem; for a general overview see [22] or [10], for results on its parameterized complexity see [13].

A question analogous to ours, asking about Graph Isomorphism on any class of $(H_1, \ldots, H_k)$-minor-free graphs, is answered completely by the fact that Graph Isomorphism is polynomially solvable on any non-trivial minor closed class [18]. Recently, the corresponding statement for topological minor free classes was also shown [9]. For the less restrictive family of *hereditary* classes, only closed under vertex deletion (i.e., classes $\mathcal{H}$-free for a possibly infinite set of graphs $\mathcal{H}$), both GI-complete and tractable cases are known: Graph Isomorphism is GI-complete on split graphs, comparability graphs, and strongly chordal graphs [23]. Graph Isomorphism is known to be polynomially solvable for interval graphs [2,15], distance hereditary graphs [17], and graphs of bounded degree [16]. For various subclasses of these polynomially solvable cases results with finer complexity analysis are available, but of course the polynomial-time solvability for these subclasses follows already from polynomial-time solvability of the mentioned larger classes. While there had been some results for circular-arc graphs, the problem is open for circular-arc graphs (see [8]). Further results, in particular on GI-completeness, can be found in [4].

Concerning our question, for one forbidden subgraph, the answer, given by Colbourn and Colbourn, can be found in a paper by Booth and Colbourn [4]: If the forbidden induced subgraph H_1 is an induced subgraph of the path P_4 on four vertices, denoted by $H_1 \leq P_4$, then Graph Isomorphism is polynomial on H_1-free graphs, otherwise it is GI-complete.

Apart from the isomorphism problem, other studies aiming at dichotomy results for algorithmic problems on graph classes characterized by two forbidden subgraphs consider the chromatic number [12] and dominating sets [14]. In the meantime it was also investigated whether such classes are well quasi-ordered [11]. (In that paper Theorem 1 also appears and the pan plays a distinguished role, see below.)

Main result. Let a graph be *basic* if it is an independent set, a clique, a $P_3 \dot\cup K_1$, or the complement of a $P_3 \dot\cup K_1$ (also called pan). If neither H_1 nor H_2 is basic, then we obtain a classification of (H_1, H_2)-free classes into polynomial and GI-complete cases, for all but a small finite number of classes. Theorem 1 justifies the terminology basic by showing that in our context forbidding a basic graph is equivalent to forbidding a complete graph. However, the case of forbidding a clique (alongside a second graph) appears to be structurally different and for a complete classification further new techniques are required.

Technical contribution. Our main technical contribution lies in establishing tractability of Graph Isomorphism on four types of (H_1, H_2)-free classes (Theorem 4 in Section 4): A structural analysis of the classes enables reductions to the polynomially-solvable case of bounded color valence [1]. This reduction appears necessary since the polynomially-solvable classes of Theorem 4 encompass all classes of graphs of bounded degree, and for these Luks' group-theoretic approach [16] (implicit in [1]) is the only known polynomial-time technique. At the core of the proof of Theorem 4 lie individualization-refinement techniques and recursive structural analysis to allow for a reduction to the bounded color valence case.

However, to put these results in context and obtain the mentioned classification, we have to refine several known results for GI-completeness on bipartite, split, and line graphs (Section 3). In particular, we arrive at a set of four graph properties, which we call *split conditions*, such that Graph Isomorphism remains complete on any class (H_1, H_2)-free unless each property is true for at least one of the two forbidden subgraphs.

Based on this characterization we can state our results in more detail: If on the one hand neither of the two forbidden subgraphs H_1 and H_2 exhibits all four split conditions, then we have a dichotomy of GI on (H_1, H_2)-free classes into polynomial and GI-complete classes; the polynomial cases are due to Theorem 4 (see Section 4) as well as tractability on cographs (i.e., P_4-free graphs) [7], GI-completeness follows by using both known results as well as our strengthened reductions (see Section 3). Suppose on the other hand H_1 and H_2 are both not basic and H_1 simultaneously fulfills all four split conditions, then our hardness and tractability results resolve all but a finite number of cases (i.e., each case is one concrete class (H_1, H_2)-free), as shown in Theorem 6 (see Section 5). For these cases Fig. 1 shows the relevant maximal graphs that adhere to all four split conditions.

2. Preliminaries

We write $H \leq G$ if the graph G contains a graph H as an induced subgraph. A graph G is H-free if $H \not\leq G$. It is $(H_1, \ldots, H_k)$-free, if it is H_i-free for all i. A graph class $\mathcal{C}$ is H-free (resp. $(H_1, \ldots, H_k)$-free) if this is true for all $G \in \mathcal{C}$. A graph class $\mathcal{C}$ is *hereditary* if it is closed under taking induced subgraphs. The class $(H_1, \ldots, H_k)$-free is the class of all $(H_1, \ldots, H_k)$-free graphs; each class $(H_1, \ldots, H_k)$-free is hereditary.

By I_t, K_t, P_t, and C_t we denote the independent set, the clique, the path, and the cycle on t vertices; $K_{1,t}$ is the claw with t leaves. By $H \dot\cup H'$ we denote the disjoint union of H and H'; we use tK_2 for the disjoint union of t graphs K_2. By $\overline{G}$ we denote the (edge) complement of G. The graph $\overline{K_2 \dot\cup I_2}$, i.e., the same as a K_4 minus one edge, is called diamond.