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Characterization and recognition of some opposition and coalition graph classes

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ABSTRACT

A graph is an opposition graph, respectively, a coalition graph, if it admits an acyclic orientation which puts the two end-edges of every chordless 4-vertex path in opposition, respectively, in the same direction. Opposition and coalition graphs have been introduced and investigated in connection to perfect graphs. Recognizing and characterizing opposition and coalition graphs are long-standing open problems. This paper gives characterizations for opposition graphs and coalition graphs on some restricted graph classes. Implicit in our arguments are polynomial time recognition algorithms for these graphs. We also give a good characterization for the so-called generalized opposition graphs.

1. Introduction and preliminaries

Chvátal [4] proposed to call a linear order $<$ on the vertex set of an undirected graph G *perfect* if the greedy coloring algorithm applied to each induced subgraph H of G gives an optimal coloring of H. Consider the vertices of H sequentially by following the order $<$ and assign to each vertex v the smallest color not used on any neighbor u of v, $u < v$. A graph is *perfectly orderable* if it admits a perfect order. Chvátal proved that $<$ is a perfect order if and only if there is no chordless path with four vertices a, b, c, d and three edges ab, bc, cd (written P_4 $abcd$) with $a < b$ and $d < c$. He also proved that perfectly orderable graphs are perfect.[1] The class of perfectly orderable graphs properly contains many important, well-known classes of perfect graphs such as chordal graphs and comparability graphs. Perfectly orderable graphs have been extensively studied in the literature; see Hoàng's comprehensive survey [11] for more information.

Recognizing perfectly orderable graphs is NP-complete [15] (see also [10]). Also, no characterization of perfectly orderable graphs by forbidden induced subgraphs is known. These facts have motivated researchers to study subclasses of perfectly orderable graphs; see, e.g., [8,11,12] and the literature given there. Observe that a linear order $<$ corresponds to an acyclic orientation by directing the edge xy from x to y if and only if $x < y$. Thus, a graph is perfectly orderable if and only if it admits an acyclic orientation such that the orientation of no chordless path P_4 is of type 0 (equivalently, the orientation of every P_4 is of type 1, 2, or 3); see Fig. 1.

One of the natural subclass of perfectly orderable graphs for which the recognition complexity, as well as an induced subgraph characterization are still unknown is the following (cf. [11,12]).

Definition 1. A graph is a *coalition graph* if it admits an acyclic orientation such that every induced P_4 $abcd$ has the end-edges ab and cd oriented in the 'same way', that is, every oriented P_4 is of type 2 or 3. Such an orientation is called a *coalition orientation*.

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[1] A graph is *perfect* if the chromatic number and the clique number are equal in every induced subgraph.

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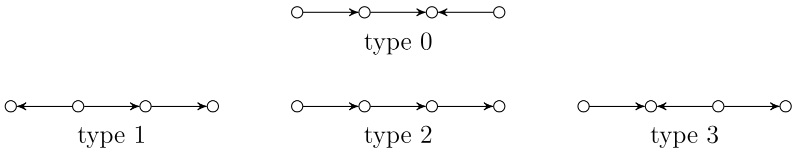

Fig. 1. Four types of oriented P_4.

Equivalently, a graph is a coalition graph if it admits a linear order $<$ on its vertex set such that every induced P_4 *abcd* has $a < b$ if and only if $c < d$. In [11], coalition graphs are called one-in-one-out graphs. Examples of coalition graphs include comparability graphs, hence all bipartite graphs.

A related graph class has been introduced by Olariu in [18]:

Definition 2. A graph is an *opposition graph* if it admits an acyclic orientation such that every induced P_4 *abcd* has the end-edges *ab* and *cd* oriented 'in opposition', that is, every oriented P_4 is of type 0 or 1. Such an orientation is called an *opposition orientation*.

Equivalently, a graph is an opposition graph if it admits a linear order $<$ on its vertex set such that every P_4 *abcd* has $a < b$ if and only if $d < c$. Olariu [18] proved that opposition graphs are perfect. He also conjectures [20] that not all opposition graphs are perfectly orderable. Examples of opposition graphs include all split graphs. The recognition and characterization problems for opposition graphs are still open.

Coalition graphs and opposition graphs have been studied in the past from the combinatorial and algorithmic point of view. The characterization and recognition problems for these graphs have been solved for a few special graph classes so far. A natural subclass of opposition graphs consists of those admitting an acyclic orientation in which every P_4 is oriented as type 1 (equivalently, every P_4 is oriented as type 0). These are called *bipolarizable graphs*, and have been characterized by (infinitely many) forbidden induced subgraphs in [9,12], and have been recognized using $O(n)$ adjacency matrix multiplications and thus in $O(n^{3.376})$ time in [8], and in $O(nm)$ time in [16]; n is the vertex number and m is the edge number of the input graph. Another subclass of opposition graphs are the so-called Welsh–Powell opposition graphs; see [21,14,17] for more information.

In a recent paper [13], bipartite opposition graphs have been characterized by (infinitely many) forbidden induced subgraphs, and have been recognized in linear time. This paper gives also characterizations for complements of bipartite graphs that are coalition or opposition graphs. It turns out that co-bipartite coalition graphs and co-bipartite opposition graphs coincide, and they are exactly the complements of bipartite permutation graphs, hence can be recognized in linear time. There is also a characterization of co-bipartite coalition/opposition graphs in terms of their bi-matrices. This characterization is similar to the one of co-bipartite perfectly orderable graphs given by Chvátal in [5], which has a close connection to a theorem in mathematical programming.

We first address in Section 2 the so-called generalized opposition graphs introduced by Chvátal; these graphs are obtained when the condition 'acyclic' in Definition 2 is dropped. This concept turns out to be useful when considering opposition graphs in certain graph classes. We give a characterization for generalized opposition graphs in terms of an auxiliary graph, which leads to a polynomial time recognition algorithm.

In Section 3 we extend the results in [13] on bipartite opposition graphs to the larger class of (gem, house)-free graphs. It turns out that inside this graph class, opposition graphs and generalized opposition graphs coincide, hence (gem, house)-free opposition graphs can be recognized in polynomial time.

In Section 4 we give a forbidden subgraph characterization for distance-hereditary opposition graphs, a subclass of (gem, house)-free opposition graphs; this result includes the subgraph characterization for tree opposition graphs found in [13].

In Section 5 we show that (gem, house, hole)-free coalition graphs can be recognized in polynomial time by modifying the auxiliary graph for generalized opposition graphs. For the smaller class of distance-hereditary coalition graphs, we give a faster recognition algorithm by showing that they are indeed comparability graphs.

Definitions and notation. We consider only finite, simple, and undirected graphs. For a graph G, the vertex set is denoted $V(G)$ and the edge set is denoted $E(G)$. For a vertex u of a graph G, the neighborhood of u in G is denoted $N_G(u)$ or simply $N(u)$ if the context is clear, and the degree of u is $\deg(u) = |N(u)|$. Write $N[u] = N(u) \cup \{u\}$. For a set U of vertices of a graph G, write $N(U) = \bigcup_{u \in U} N(u) \setminus U$ and $N[U] = N(U) \cup U$. The subgraph of G induced by U is denoted $G[U]$. If u is a vertex of a graph G, then $G - u$ is $G[V(G) \setminus \{u\}]$.

For $\ell \geq 1$, let P_ℓ denote a chordless path with ℓ vertices and $\ell - 1$ edges, and for $\ell \geq 3$, let C_ℓ denote a chordless cycle with ℓ vertices and ℓ edges. We write P_ℓ $u_1u_2 \ldots u_\ell$ and C_ℓ $u_1u_2 \ldots u_\ell u_1$, meaning the chordless path with vertices $u_1, u_2, \ldots, u_\ell$ and edges u_iu_{i+1}, $1 \leq i < \ell$, respectively, the chordless cycle with vertices $u_1, u_2, \ldots, u_\ell$ and edges u_iu_{i+1}, $1 \leq i < \ell$, and $u_\ell u_1$. The edges u_1u_2 and $u_{\ell-1}u_\ell$ of the path P_ℓ ($\ell \geq 3$) are the *end-edges* and the other edges are the *mid-edges* of the path, while the vertices u_1 and u_ℓ are the *end-vertices* and the other vertices are the *mid-vertices* of the path. In this paper, all paths P_ℓ and all cycles C_ℓ will always be induced.