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Algorithmic aspects of open neighborhood location–domination in graphs

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ABSTRACT

A set $D \subseteq V$ of a graph $G = (V, E)$ is called an open neighborhood locating–dominating set (OLD-set) if (i) $N_G(v) \cap D \neq \emptyset$ for all $v \in V$, and (ii) $N_G(u) \cap D \neq N_G(v) \cap D$ for every pair of distinct vertices $u, v \in V$. Given a graph $G = (V, E)$, the Min OLD-set problem is to find an OLD-set of minimum cardinality. Given a graph $G = (V, E)$ and a positive integer k, the Decide OLD-set problem is to decide whether G has an OLD-set of cardinality at most k. The Decide OLD-set problem is known to NP-complete for general graphs. In this paper we extend the NP-completeness result of the Decide OLD-set problem by showing that it remains NP-complete for bipartite graphs, planar graphs, split graphs and doubly chordal graphs. We prove that the Decide OLD-set problem can be solved in linear time for bounded tree-width graphs. We, then, propose a linear time algorithm for the Min OLD-set problem in trees. We also propose a $(2 + 3 \ln \Delta)$-approximation algorithm for the Min OLD-set problem and show that the Min OLD-set problem cannot be approximated within $\frac{1}{2}(1 - \epsilon) \ln |V|$ for any $\epsilon > 0$ unless NP $\subseteq$ DTIME($|V|^{O(\log \log |V|)}$). Finally, we prove that the Min OLD-set problem is APX-complete for bipartite graphs of maximum degree 3.

1. Introduction

For a graph $G = (V, E)$, the sets $N_G(v) = \{u \in V(G) \mid uv \in E\}$ and $N_G[v] = N_G(v) \cup \{v\}$ denote the *open neighborhood* and *closed neighborhood* of a vertex v, respectively. A set $D \subseteq V$ is called a *dominating set* of $G = (V, E)$ if $N_G(v) \cap D \neq \emptyset$ for every vertex $v \in V \setminus D$. The Min Dom Set problem is to find a dominating set of minimum cardinality. The concepts of domination and its variations are widely studied (see [11,12]). A *total dominating set* of a graph G is a set $D \subseteq V(G)$ such that for every vertex $v \in V(G)$, $N_G(v) \cap D \neq \emptyset$. The Min Total Dom Set problem is to find a total dominating set of minimum cardinality. The Min Total Dom Set problem is also a well studied problem in graph theory, as can be seen in the recent survey paper [13].

Consider a graph, that models a network. We want to place the sensors at the vertices, so as to uniquely identify a fault. It is assumed that a sensor placed at vertex v can detect a fault only if the fault is at a vertex location adjacent to v. To detect the fault uniquely, we need to place the sensors at all the vertices of an open neighborhood locating–dominating set of the graph, representing the network.

For a graph G, a total dominating set D is called an *open neighborhood locating–dominating set* (OLD-set) if for every pair of distinct vertices $u, v \in V$, $N_G(u) \cap D \neq N_G(v) \cap D$. Note that not every graph admits an OLD-set. The *minimum open*

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neighborhood locating–dominating set problem (MIN OLD-SET problem) is to find an OLD-set of minimum cardinality of a graph that admits an OLD-set. The *open location–domination number* of G is the cardinality of a minimum OLD-set of G, and is denoted by $OLD(G)$. Given a positive integer k and a graph $G = (V, E)$, the DECIDE MIN OLD-SET problem is to decide whether G has an OLD-set of cardinality at most k. The concept of open location–domination number was introduced by Seo and Slater [18]. The DECIDE MIN OLD-SET problem is known to be NP-complete for general graphs [18]. In this paper, we study the algorithmic aspects of the MIN OLD-SET problem. The main contributions are as follows.

- We extend the NP-completeness result of the DECIDE OLD-SET problem by showing that it remains NP-complete for bipartite graphs, planar graphs, split graphs and doubly chordal graphs.
- We prove that the DECIDE OLD-SET problem can be solved in linear time for bounded tree-width graphs. We, then, propose a linear time algorithm for the MIN OLD-SET problem in trees.
- We propose a $(2 + 3\ln\Delta)$-approximation algorithm for the MIN OLD-SET problem and show that the MIN OLD-SET problem cannot be approximated within $\frac{1}{2}(1-\epsilon)\ln|V|$ for any $\epsilon > 0$ unless NP $\subseteq$ DTIME$(|V|^{O(\log\log|V|)})$.
- We prove that the MIN OLD-SET problem is APX-complete for bipartite graphs of maximum degree 3.

The paper is organized as follows. In Section 2, some pertinent definitions and some preliminary results are discussed. In Section 3, we have shown that the DECIDE MIN OLD-SET problem remains NP-complete for bipartite graphs, planar graphs, split graphs and doubly chordal graphs. In Section 4, we have shown the graph classes where the MIN DOM SET problem and the MIN OLD-SET problem differ in complexity. In Section 5, we prove that the DECIDE MIN OLD-SET problem is linear time solvable for bounded tree-width graphs. In Section 6, we propose a linear time algorithm to solve the MIN OLD-SET problem for trees. An approximation algorithm for computing a minimum OLD-set of a graph, is proposed in Section 7. In this section, we have also proved an approximation hardness result for computing a minimum OLD-set of a graph. In addition, the MIN OLD-SET problem is shown to be APX-complete in this section. Finally, Section 8 concludes the paper.

2. Preliminaries

In a graph $G = (V, E)$, the *degree* of a vertex v is $|N_G(v)|$ and is denoted by $d_G(v)$. If $d_G(v) = 1$, then v is called a *pendant vertex*. The vertex adjacent to a pendant vertex is called a *support vertex*. Let $\Delta(G)$ and $\delta(G)$ denote the *maximum degree* and *minimum degree* of a graph G, respectively. For a set $S \subseteq V$ of the graph $G = (V, E)$, the subgraph of G *induced* by S is defined as $G[S] = (S, E_S)$, where $E_S = \{xy \in E | x, y \in S\}$. If $G[C]$, where $C \subseteq V$, is a complete subgraph of G, then C is called a *clique* of G. A set $S \subseteq V$ is an *independent set* if $G[S]$ has no edge. A graph $G = (V, E)$ is said to be *bipartite* if V can be partitioned into two disjoint sets X and Y such that every edge of G joins a vertex in X to a vertex in Y. Such a partition (X, Y) of V is called a *bipartition*. A bipartite graph with bipartition (X, Y) of V is denoted by $G = (X, Y, E)$. Let n and m denote the number of vertices and number of edges of G, respectively. A graph G is said to be a *chordal graph* if every cycle in G of length at least four has a *chord*, that is, an edge joining two non-consecutive vertices of the cycle. A chordal graph $G = (V, E)$ is a *split graph* if V can be partitioned into two sets I and C such that C is a clique and I is an independent set. A vertex $v \in V(G)$ is a *simplicial* vertex of G if $N_G[v]$ is a clique of G. An ordering $\alpha = (v_1, v_2, \ldots, v_n)$ is a *perfect elimination ordering* (PEO) of G if v_i is a simplicial vertex of $G_i = G[\{v_i, v_{i+1}, \ldots, v_n\}]$ for all i, $1 \le i \le n$. It is characterized that a graph G is chordal if and only if it has a PEO [9]. A vertex $u \in N_G[v]$ is a *maximum neighbor* of v in G if $N_G[w] \subseteq N_G[u]$ for all $w \in N_G[v]$. A vertex v in G is called *doubly simplicial* if it is simplicial and has a maximum neighbor in G. An ordering $\alpha = \{v_1, v_2, \ldots, v_n\}$ of vertices of G is a *doubly perfect elimination ordering* (DPEO) if v_i is doubly simplicial vertex in the induced subgraph $G[\{v_i, v_{i+1}, \ldots, v_n\}]$ for each i, $1 \le i \le n$. A graph is *doubly chordal* if it admits a doubly perfect elimination ordering (DPEO) [4]. A graph G is called a *planar graph* if it can be drawn on the plane in such a way that no two edges cross each other except at a vertex. Such a drawing is called a *planar embedding* of the planar graph.

Let G be a graph, T be a tree and $\mathcal{V}$ be a family of vertex sets $V_t \subseteq V(G)$ indexed by the vertices t of T. The pair $(T, \mathcal{V})$ is called a *tree-decomposition* of G if it satisfies the following three conditions:

1. $V(G) = \bigcup_{t \in V(T)} V_t$,
2. for every edge $e \in E(G)$ there exists a $t \in V(T)$ such that both ends of e lie in V_t,
3. $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ whenever $t_1, t_2, t_3 \in V(T)$ and t_2 is on the path in T from t_1 to t_3.

The *width* of (T, ν) is the number $\max\{|V_t| - 1 : t \in T\}$, and the *tree-width* $tw(G)$ of G is the least width of any tree-decomposition of G [8].

In the rest of the paper, we only consider simple connected graphs with at least two vertices unless otherwise mentioned specifically. The following theorem gives a necessary and sufficient condition for a graph to have an OLD-set.

Theorem 2.1 ([18]). *A graph G has an OLD-set if and only if for any two distinct vertices $u, v \in V(G)$, we have $N_G(u) \neq N_G(v)$.*

From the above theorem, it follows that if a tree has an OLD-set, then no support vertex is adjacent to two or more pendant vertices.