ARTICLE IN PRESS
Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Graphs of edge-intersecting non-splitting paths in a tree: Representations of holes—Part I[☆,☆☆]

Arman Boyacı[a], Tınaz Ekim[a], Mordechai Shalom[a,b,*], Shmuel Zaks[c,d]

[a] Department of Industrial Engineering, Bogazici University, Istanbul, Turkey
[b] TelHai Academic College, Upper Galilee, 12210, Israel
[c] Department of Computer Science, Technion, Haifa, Israel
[d] Catedra de Excelencia of Universidad Carlos III de Madrid, Departmento de Ingenieria Telematica, Spain

ARTICLE INFO

Article history:
Received 12 November 2013
Received in revised form 6 July 2015
Accepted 19 July 2015
Available online xxxx

Keywords:
Intersection graphs
EPT graphs

ABSTRACT

Given a tree and a set $\mathcal{P}$ of non-trivial simple paths on it, $\textsc{Vpt}(\mathcal{P})$ is the VPT graph (i.e. the vertex intersection graph) of the paths $\mathcal{P}$ of the tree T, and $\textsc{Ept}(\mathcal{P})$ is the EPT graph (i.e. the edge intersection graph) of $\mathcal{P}$. These graphs have been extensively studied in the literature. Given two (edge) intersecting paths in a graph, their *split vertices* are the vertices having degree at least 3 in their union. A pair of (edge) intersecting paths is termed *non-splitting* if they do not have split vertices (namely if their union is a path).

In this work, motivated by an application in all-optical networks, we define the graph $\textsc{Enpt}(\mathcal{P})$ of edge-intersecting non-splitting paths of a tree, termed the ENPT graph, as the (edge) graph having a vertex for each path in $\mathcal{P}$, and an edge between every pair of paths that are both edge-intersecting and non-splitting. A graph G is an ENPT graph if there is a tree T and a set of paths $\mathcal{P}$ of T such that $G = \textsc{Enpt}(\mathcal{P})$, and we say that $\langle T, \mathcal{P}\rangle$ is a *representation* of G. We first show that cycles, trees and complete graphs are ENPT graphs.

Our work follows the lines of Golumbic and Jamison's research (Golumbic and Jamison, 1985) in which they defined the EPT graph class, and characterized the representations of chordless cycles (holes). It turns out that ENPT holes have a more complex structure than EPT holes. In our analysis, we assume that the EPT graph corresponding to a representation of an ENPT hole is given. We also introduce three assumptions (P1), (P2), (P3) defined on EPT, ENPT pairs of graphs. In this Part I, using the results of Golumbic and Jamison as building blocks, we characterize (a) EPT, ENPT pairs that satisfy (P1)–(P3), and (b) the unique minimal representation of such pairs.

1. Introduction

Given a tree T and a set $\mathcal{P}$ of non-trivial simple paths in T, the Vertex Intersection Graph of Paths in a Tree (VPT) and the Edge Intersection Graph of Paths in a Tree (EPT) of $\mathcal{P}$ are denoted by $\textsc{Vpt}(\mathcal{P})$ and $\textsc{Ept}(\mathcal{P})$, respectively. Both graphs have $\mathcal{P}$

☆ A preliminary version of this paper appeared in Boyacı et al. (2013) [2].

☆☆ This work was supported in part by the Israel Science Foundation grant No. 1249/08, by TUBITAK PIA BOSPHORUS Grant No. 111M303, by the TUBITAK 2221 Programme, Bogazici University Scientific Research Fund BAP grant no 6461D.

* Corresponding author at: TelHai Academic College, Upper Galilee, 12210, Israel.
E-mail addresses: arman.boyaci@boun.edu.tr (A. Boyacı), tinaz.ekim@boun.edu.tr (T. Ekim), cmshalom@telhai.ac.il (M. Shalom), zaks@cs.technion.ac.il (S. Zaks).

http://dx.doi.org/10.1016/j.dam.2015.07.024
0166-218X/

as vertex set. $\mathrm{VPT}(\mathcal{P})$ (resp. $\mathrm{EPT}(\mathcal{P})$) contains an edge between two vertices if the corresponding two paths intersect in at least one vertex (resp. edge). A graph G is VPT (resp. EPT) if there exist a tree T and a set $\mathcal{P}$ of non-trivial simple paths in T such that G is isomorphic to $\mathrm{VPT}(\mathcal{P})$ (resp. $\mathrm{EPT}(\mathcal{P})$). In this case we say that $\langle T, \mathcal{P}\rangle$ is a VPT (resp. an EPT) representation of G.

In this work we focus on edge intersections of paths. The graph of edge-intersecting and non-splitting paths of a tree (ENPT) of a given representation $\langle T, \mathcal{P}\rangle$ denoted by $\mathrm{ENPT}(\mathcal{P})$, has a vertex v for each path P_v of $\mathcal{P}$ and two vertices u, v of $\mathrm{ENPT}(\mathcal{P})$ are adjacent if the paths P_u and P_v edge-intersect and do not split (that is, their union is a path). A graph G is an ENPT graph if there is a tree T and a set of paths $\mathcal{P}$ of T such that G is isomorphic to $\mathrm{ENPT}(\mathcal{P})$. We note that $\mathrm{EPT}(\mathcal{P}) = \mathrm{ENPT}(\mathcal{P})$ is an interval graph whenever T is a path. Therefore the class ENPT includes all interval graphs.

1.1. Applications

EPT and VPT graphs have applications in communication networks. Consider a communication network of a tree topology T. The message routes to be delivered in this communication network are paths on T. Two paths conflict if they both require to use the same link (node). This conflict model is equivalent to an EPT (a VPT) graph. Suppose we try to find a schedule for the messages such that no two messages sharing a link (node) are scheduled in the same time interval. Then a vertex coloring of the EPT (VPT) graph corresponds to a feasible schedule on this network.

EPT graphs also appear in all-optical telecommunication networks. The so-called Wavelength Division Multiplexing (WDM) technology can multiplex different signals onto a single optical fiber by using different wavelength ranges of the laser beam [5,18]. WDM is a promising technology enabling us to deal with the massive growth of traffic in telecommunication networks, due to applications such as video-conferencing, cloud computing and distributed computing [6]. A stream of signals traveling from its source to its destination in optical form is termed a *lightpath*. A lightpath is realized by signals traveling through a series of fibers, on a certain wavelength. Specifically, Wavelength Assignment problems (WLA) are a family of path coloring problems that aim to assign wavelengths (i.e. colors) to lightpaths, so that no two lightpaths with a common edge receive the same wavelength and certain objective function (depending on the problem) is minimized.

Traffic Grooming is the term used for the combination of several low (i.e. sub-wavelength) capacity requests (modeled by paths of a network) into one lightpath (modeled by a path or cycle of the network) using Time Division Multiplexing (TDM) technology [9]. In this context a set of paths can be combined into one lightpath, as long as they satisfy the following two conditions:

- The load condition: On any given fiber, at most g requests can use the same lightpath, where g is an integer termed *the grooming factor*.
- The no-split condition: a lightpath (i.e. the union of the requests using the lightpath) constitutes a path or a cycle of the network.

Clearly, the second condition cannot be tested in the EPT model. For this reason we introduce the ENPT graphs that provide the required information.

Readers unfamiliar with optical networks may consider the following analogous problem in transportation. Consider a set of transportation requests modeled by paths, and trucks traveling along paths or cycles. Trucks are able to load and drop items during their journey as long as at any given time their load does not exceed their capacity. The no-split condition reflects the fact that a truck has to follow a path or a cycle.

By the no-split condition, a (feasible) traffic grooming corresponds to a vertex coloring of the graph $(V(\mathrm{EPT}(\mathcal{P})), E(\mathrm{EPT}(\mathcal{P})) \setminus E(\mathrm{ENPT}(\mathcal{P})))$. Moreover, by the load condition, every color class induces a sub-graph of $\mathrm{EPT}(\mathcal{P})$ with clique number at most g. Therefore, it makes sense to analyze the structure of these graph pairs, i.e. the two graphs $\mathrm{EPT}(\mathcal{P})$ and $\mathrm{ENPT}(\mathcal{P})$ defined on the same set of vertices.

Under this setting one can consider various objective functions such as:

- Minimize the number of wavelengths / trucks: When the number of wavelengths (resp. trucks) is scarce, one aims to minimize this number. We note that when the parameter g is sufficiently big (i.e. $g = \infty$) the problems boils down to the minimum vertex coloring problem.
- Minimize the number of regenerators / total distance traveled: The signal traveling on a lightpath has to be regenerated along its way, implying a regeneration cost roughly proportional to its length. Similarly, a truck incurs operation expenses proportional to the distance it travels.

  In order to model this problem, one has to assign weights to the ENPT edges indicating the length of the overlap of two non-splitting requests. The gains obtained from putting two requests in the same lightpath (truck) is exactly the weight of the corresponding ENPT edge. Therefore, an optimal solution corresponds to a partition of the vertices into paths of $\mathrm{ENPT}(\mathcal{P})$ (without a chord from $\mathrm{EPT}(\mathcal{P})$ and with a maximum clique size of at most g), such that the sum of the weights of the edges of these paths is maximized.

1.2. Related work

EPT and VPT graphs have been extensively studied in the literature. Although VPT graphs can be characterized by a fixed number of forbidden subgraphs [17], it is shown that EPT graph recognition is NP-complete [11]. Edge intersection and vertex