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The set chromatic number of random graphs

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ABSTRACT

In this paper we study the set chromatic number of a random graph $\mathcal{G}(n, p)$ for a wide range of $p = p(n)$. We show that the set chromatic number, as a function of p, forms an intriguing zigzag shape.

1. Introduction

A *proper colouring* of a graph is a labelling of its vertices with colours such that no two vertices sharing the same edge have the same colour. A colouring using at most k colours is called a *proper* k*-colouring*. The smallest number of colours needed to colour a graph G is called its *chromatic number*, and it is denoted by $\chi(G)$.

In this paper we are concerned with another notion of colouring, first introduced by Chartrand, Okamoto, Rasmussen and Zhang [1]. For a given (not necessarily proper) k-colouring $c : V \to [k]$ of the vertex set of $G = (V, E)$, let

$$C(v) = \{c(u) : uv \in E\}$$

be the *neighbourhood colour set* of a vertex v. (In this paper, $[k] := \{1, 2, \dots, k\}$.) The colouring c is a *set colouring* if $C(u) \neq C(v)$ for every pair of adjacent vertices in G. The minimum number of colours, k, required for such a colouring is the *set chromatic number* $\chi_s(G)$ of G. One can show that

$$\log_2 \chi(G) + 1 \leq \chi_s(G) \leq \chi(G). \tag{1}$$

Indeed, the upper bound is trivial, since any proper colouring c is also a set colouring: for any edge uv, $N(u)$, the neighbourhood of u, contains $c(v)$ whereas $N(v)$ does not. On the other hand, suppose that there is a set colouring using at most k colours. Since there are at most 2^k possible neighbourhood colour sets, one can assign a unique colour to each set obtaining a proper colouring using at most 2^k colours. We get that $\chi(G) \leq 2^{\chi_s(G)}$, or equivalently, $\chi_s(G) \geq \log_2 \chi(G)$. With slightly more work, one can improve this lower bound by 1 (see [8]), which is tight (see [2]).

Let us recall a classic model of random graphs that we study in this paper. The *binomial random graph* $\mathcal{G}(n, p)$ is the random graph G with vertex set $[n]$ in which every pair $\{i, j\} \in \binom{[n]}{2}$ appears independently as an edge in G with probability p. Note that $p = p(n)$ may (and usually does) tend to zero as n tends to infinity.

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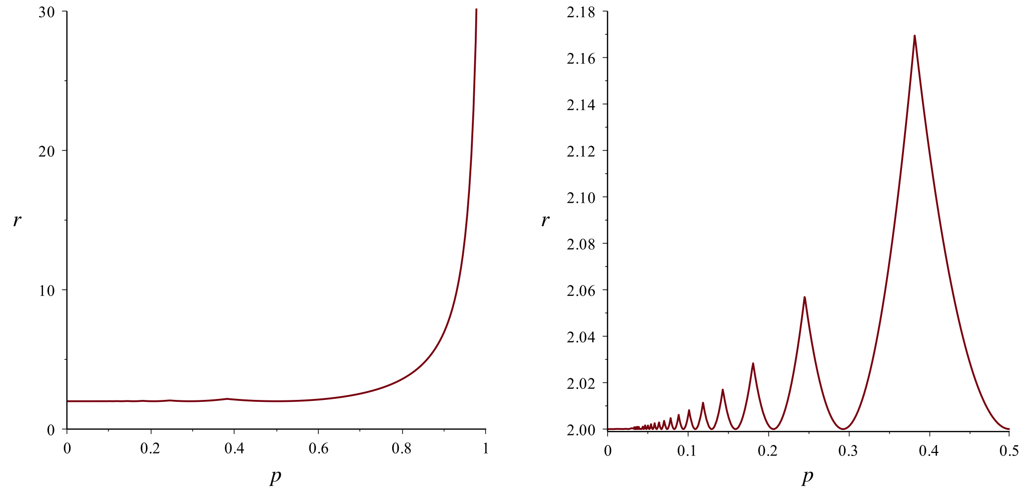

Fig. 1. The function $r = r(p)$ for $p \in (0, 1)$ and $p \in (0, 1/2]$, respectively.

All asymptotics throughout are as $n \to \infty$ (we emphasize that the notations $o(\cdot)$ and $O(\cdot)$ refer to functions of n, not necessarily positive, whose growth is bounded). We also use the notations $f \ll g$ for $f = o(g)$ and $f \gg g$ for $g = o(f)$. We say that an event in a probability space holds *asymptotically almost surely* (or *a.a.s.*) if the probability that it holds tends to 1 as n goes to infinity. Since we aim for results that hold a.a.s., we will always assume that n is large enough. We often write $\mathcal{G}(n, p)$ when we mean a graph drawn from the distribution $\mathcal{G}(n, p)$. For simplicity, we will write $f(n) \sim g(n)$ if $f(n)/g(n) \to 1$ as $n \to \infty$ (that is, when $f(n) = (1 + o(1))g(n)$). Finally, we use lg to denote logarithms with base 2 and log to denote natural logarithms.

Before we state the main result of this paper, we need a few definitions that we will keep using throughout the whole paper. For a given $p = p(n)$ satisfying

$$p \geq \frac{4}{\log 2} \cdot \frac{(\log n)(\log\log n)}{n} \quad \text{and} \quad p \leq 1 - \varepsilon$$

for some $\varepsilon > 0$, let

$$s = s(p) = \min\left\{[(1-p)^{\ell}]^2 + [1-(1-p)^{\ell}]^2 : \ell \in \mathbb{N}\right\},$$

and let ℓ_0 be a value of ℓ that achieves the minimum (ℓ_0 can be assigned arbitrarily if there are at least two such values). We will show in Section 3 that

$$\ell_0 \in \left\{ \left\lfloor \frac{\log(1/2)}{\log(1-p)} \right\rfloor, \left\lceil \frac{\log(1/2)}{\log(1-p)} \right\rceil \right\}, \tag{2}$$

and that

$$\frac{1}{2} \leq s(p) \leq \frac{1+p^2}{2}. \tag{3}$$

If p is a constant, then $r = r(p)$ is defined such that $n^2 s^{r \lg n} = 1$, that is,

$$r = r(p) = \frac{2}{\lg(1/s)}. \tag{4}$$

Observe that r tends to infinity as $p \to 1$ and undergoes a "zigzag" behaviour as a function of p (see Fig. 1). The reason for such a behaviour is, of course, that the function s is not monotone (see Fig. 2). Furthermore, observe that for each $p = 1-(1/2)^{1/k}$, where k is a positive integer, $\ell_0 = k$, $s = 1/2$, and $r = 2$.

Now we state the main result of the paper.

Theorem 1.1. *Suppose that $p = p(n)$ is such that*

$$p \gg (\log n)^2(\log\log n)^2/n \quad \textit{and} \quad p \leq 1 - \varepsilon,$$

for some $\varepsilon \in (0, 1)$. Let $G \in \mathcal{G}(n, p)$. Then, the following holds a.a.s.

(i) *If p is a constant, then*

$$\chi_s(G) \sim r \lg n.$$

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